Ratio Test & Root Test

Ratio Test

Given the series $\sum_{n=1}^{\infty} a_n$.

(1) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series is absolutely convergent.

(2) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series is divergent.

(3) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then the series might converge or diverge. No conclusion can be made, and use another test.

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{n3^n}{4^{n-1}}$ converges absolutely.

Solution. Let $a_n = \frac{n3^n}{4^{n-1}}$. Then, $a_{n+1} = \frac{(n+1)3^{n+1}}{4^{(n+1)-1}} = \frac{(n+1)3^{n+1}}{4^n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)3^{n+1}}{4^n}}{\frac{n3^n}{4^{n-1}}}\right| = \lim_{n\to\infty}\left|\left(\frac{(n+1)3^{n+1}}{4^n}\right)\left(\frac{4^{n-1}}{n3^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{3(n+1)}{4n}\right| = \lim_{n\to\infty}\left|\frac{3(1+1/n)}{4}\right| = \frac{3}{4} < 1$$

By the Ratio Test, $\sum_{n=1}^{\infty} \frac{n3^n}{4^{n-1}}$ is absolutely convergent.

Root Test

Given the series $\sum_{n=1}^{\infty} a_n$.

(1) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series is absolutely convergent.

(2) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series is divergent.

(3) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then the series might converge or diverge. No conclusion can be made, and use another test (do not use the Ratio Test because L will be 1 again.)

Example. Determine if the series $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ absolutely converges.

Solution. Let $a_n = \frac{n^n}{3^{1+3n}}$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n^n}{3^{1+3n}}\right|} = \lim_{n\to\infty} \left(\frac{n^n}{3^{1+3n}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3^{1/n+3}} = \infty$$

By the Root Test, $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$ diverges.