Limits as x approaches a Real Number

Limits

Roughly speaking, finding the limit involves examining the behaviour of a function $f(x)$ as x approaches a real number a that may or may not be in the domain of f, that is,

$$\lim_{x \to a} f(x) = L, \text{ where } L \text{ is a real number,}$$

then it is said that the limit exists. Otherwise, the limit does not exist.

> **It is important to remember that limits describe the behaviour of a function near a particular point, but not necessarily at the point itself!**

Computation of Limits in Some Cases

<u>Case 1.</u> Direct substitution rule.

Example. Find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

Solution. By substituting $x = -2$,

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

<u>Case 2.</u> If the fraction is of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$, then, sometimes, there is HOPE!

Try to manipulate $f(x)$ by rationalizing, factoring, etc. in order to cancel. (Alternatively, L'Hôpital's Rule can be used.)

Example. Compute $\lim_{x\to 2}\frac{x-4}{\sqrt{x}-2}$.

Solution. The fraction has the form $\frac{0}{0}$. HOPE!

By using the difference of squares,

$$\lim_{x\to 4}\frac{x-4}{\sqrt{x}-2}=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-\sqrt{2}}$$
$$=\lim_{x\to 4}\sqrt{x}+\sqrt{2}=\sqrt{4}+\sqrt{2}$$
$$=2+\sqrt{2}$$

<u>Case 3.</u> If the fraction is of the form $\frac{1}{0}$, then the limit does not exist. (Note that 1 in the numerator can be any non-zero number.)

Example. Evaluate $\lim_{x\to -3}\frac{x^2-x+12}{x+3}$, if possible.

Solution. Note that $f(x)$ cannot be reduced.

The limit of the numerator is $\lim_{x\to -3}(x^2-x+12)=24$, which is not equal to zero.

The limit of the denominator is $\lim_{x\to -3} x+3=0$.

Thus, the limit does not exist.

Squeeze Theorem Suppose that $g(x)\le f(x)\le h(x)$ for all x in an open interval containing a, except possibly at $x=a$ itself and that $\lim_{x\to a}g(x)=\lim_{x\to a}h(x)=L$. Then, $\lim_{x\to a}f(x)=L$.

Example. Evaluate $\lim_{x\to 0} x^2 e^{\sin(1/x)}$.

Solution. Using the fact that $-1 \le \sin(x) \le 1$ in order to create an inequality,

$$-1 \le \sin(1/x) \le 1$$
$$e^{-1} \le e^{\sin(1/x)} \le e^{1}$$
$$x^2 e^{-1} \le x^2 e^{\sin(1/x)} \le x^2 e$$

Since $x^2 e^{-1} \le x^2 e^{\sin(1/x)} \le x^2 e$ and $\lim_{x\to 0} x^2 e^{-1} = \lim_{x\to 0} x^2 e = 0$, $\lim_{x\to 0} x^2 e^{\sin(1/x)} = 0$ by the Squeeze Theorem. (Note that the limit cannot be evaluated by direct substitution as the limit of the exponential function, $e^{\sin(1/x)}$, does not exist.