Sequences

A sequence is a list of numbers written in a definite order denoted as:

$$a_1, a_2, \ldots, a_n, \ldots = \{a_n\},$$

where $a_1$ is the first term of the sequence, $a_2$ is the second term and in general, $a_n$ is the nth term. (Note that often, unless stated otherwise, it is assume that the values of $n$ start with $n = 1$).

For example, $5, 10, 15, 20, 25 \ldots = \{5n\} = \{a_n\}$ or

$$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81} \ldots = \left\{\frac{1}{(-3)^n}\right\}_n^{\infty}$$

If a sequence $\{a_n\}$ has a limit, that is $\lim_{n \to \infty} a_n = L$ where $L$ is a real number, then it is said to be convergent, and converges to $L$.

If the limit of the sequence does not exist, then the sequence is said to be divergent.

For instance, the sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$, that is,

$$\lim_{n \to \infty} r^n = \begin{cases} 0, & \text{if } -1 < r < 1 \\ 1, & \text{if } r = 1 \end{cases}, \text{ and divergent otherwise.}$$

Example. Determine if each sequence is convergent or divergent.

(a) $\{a_n\} = \{3 + e^{-n}\} \quad$ (b) $\{a_n\} = \{(-0.5)^n\}$
Solution.  

(a) \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} (3 + e^{-n}) = 3 \). Convergent.

(b) \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} (-0.5)^n = 0 \). Convergent.

(c) \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{3 + 5n} = \lim_{n \to \infty} \frac{n}{3/n + 5} = +\infty \). Divergent.

(d) Note that \( \{a_n\} = \{(-1)^n\} = -1, 1, -1, 1, ... \)
\( \lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n \) does not exist. Divergent.