Integral Test

Suppose $f$ is continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then, the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) \, dx$ is convergent.

Remember: By the Integral Test, the following result can be proven.

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) is divergent since it is a p-series with $p = 1$.

Example. Determine whether the series $\sum_{n=1}^{\infty} ne^{-n^2}$ diverges or converges.

Solution. Let $f(x) = xe^{-x^2}$. The function $f(x)$ is continuous, positive, and decreasing, since $f'(x) = e^{-x^2} (1 - 2x) < 0$ for all $x$ values in $[1, \infty)$. By the integral test, (using integration by substitution)

$$\int_1^{\infty} xe^{-x^2} \, dx = \lim_{t \to \infty} \int_1^{t} xe^{-x^2} \, dx = \lim_{t \to \infty} \frac{-e^{-x^2}}{2} \bigg|_1^t$$

$$= \lim_{t \to \infty} \left( -\frac{e^{-t^2}}{2} - \left( -\frac{e^{-1}}{2} \right) \right) = \frac{e^{-1}}{2} = \frac{1}{2e}$$
since \( \lim_{x \to \infty} e^{-x} = 0 \). Thus, the series \( \sum_{n=1}^{\infty} ne^{-n^2} \) converges.