Absolute Convergence & Conditional Convergence

Absolutely Convergent Series

Given a series $\sum_{n=1}^{\infty} a_n$, consider the corresponding series $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \ldots$. The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem: If a series is absolutely convergent, then it is convergent.

Note that the reserve is not true (the series in Example (a) below is convergent, but not absolutely convergent).

Conditionally Convergent Series

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it is convergent, but not absolutely convergent.

Example. Determine if each series is absolutely convergent or conditionally convergent.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}$

Solution. (a) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$. This is a p-series with $p = 1$, and thus, diverges.

But, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test. Thus, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is conditionally convergent.

(b) $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{3^n}\right| = \sum_{n=1}^{\infty}\frac{1}{3^n}$. This is a geometric series with $r = \frac{1}{3} < 1$ which converges. Thus, $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3^n}$ is absolutely convergent (and thus, convergent as well).