Limit as x approaches Infinity

Limits as $x \to \pm\infty$

When the values of the function $f(x)$ approach the number L as x increases without a bound (meaning that the values of x can be chosen to be larger than any real number chosen),

$$\lim_{x \to +\infty} f(x) = L \ .$$

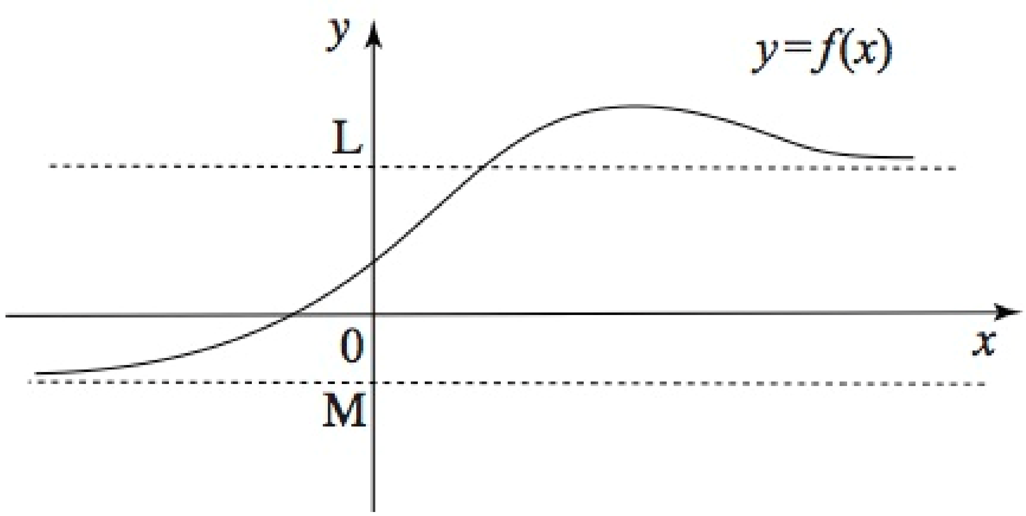
Likewise,

$$\lim_{x \to -\infty} f(x) = M$$

when the functional values $f(x)$ approach the number M as x decreases without a bound.

Horizontal Asymptotes

Geometrically, $\lim_{x \to +\infty} f(x) = L$ means that the graph of $f(x)$ approaches the horizontal line $y = L$ at infinity, while $\lim_{x \to -\infty} f(x) = M$ means that the graph of $f(x)$ approaches the horizontal line $y = M$ at negative infinity.



The lines $y = L$ and $y = M$ are called horizontal asymptotes.

Note that to find horizontal asymptotes, both the limit at $+\infty$ and at $-\infty$ must be checked.

Reciprocal Power Rules

If A and p are constants with $p > 0$ and x^p is defined for all x, then

$$\lim_{x \to \infty} \frac{A}{x^p} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{A}{x^p} = 0 .$$

Computation of Limits when $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{p(x)}{q(x)}$

Step 1. Divide each term in $f(x)$ by the highest power of x^p that appears in the denominator. (Alternatively, divide by the highest power in the numerator, or by the highest power overall in $f(x)$.)

Step 2. Compute $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ using algebraic properties of limits and the reciprocal power rules.

Example. Evaluate $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$.

Solution. The highest power in the denominator is x^2.
Divide the numerator and denominator by x^2 to get:

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x \to \infty} \frac{3 - 1/x - 2/x^2}{5 + 4/x + 1/x^2} = \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}$$

Thus, the graph of $f(x) = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$ has a horizontal asymptote at $y = \frac{3}{5}$. Note that the limit at $-\infty$ gives the same answer.

Example. Find $\lim_{r\to-\infty}\dfrac{r^4-r^2+1}{r^5+r^3-r}$.

Solution. $$\lim_{r\to-\infty}\frac{r^4-r^2+1}{r^5+r^3-r}=\lim_{r\to-\infty}\frac{1/r-1/r^3+1/r^5}{1+1/r^2-1/r^4}=\frac{0-0+0}{1+0-0}=0$$

There is a horizontal asymptote at $y=0$. Note that the limit at $+\infty$ gives the same answer.

Example. Find $\lim_{x\to+\infty}\dfrac{x^2+x}{3-x}$.

Solution. $$\lim_{x\to+\infty}\frac{x^2+x}{3-x}=\lim_{x\to+\infty}\frac{x+1}{3/x-1}=-\infty$$

Thus, the graph of $f(x)=\dfrac{x^2+x}{3-x}$ has no horizontal asymptote at $+\infty$.

Note that to find the horizontal asymptotes, the limit at $-\infty$ must be checked as well.