Definition of Continuity

A function \( f(x) \) is continuous at a point \( x = a \) if and only if the following conditions are satisfied:

1. \( f(a) \) exists (i.e., \( a \) lies in the domain of \( f \))
2. \( \lim_{x \to a} f(x) \) exists (\( f \) has a limit as \( x \to a \), i.e., the limit is a real number)
3. \( \lim_{x \to a} f(x) = f(a) \) (the limit equals the function value)

If one or more of these conditions fail, then \( f(x) \) is not continuous at a point \( x = a \).

Example. Consider the function \( f(x) = \begin{cases} x^3 + 2x^2 - 8x & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases} \)

(a) Is \( f(x) \) continuous at \( x = -2 \)?
(b) Does \( \lim_{x \to 2} f(x) \) exist?
(c) Is \( f(x) \) continuous at \( x = 2 \)?

Solution.

(a) \( f(-2) = \frac{(-2)^3 + 2(-2)^2 - 8(-2)}{(-2)^2 - 4} = \frac{16}{0} \). Thus, \( f(-2) \) is not defined, and hence, \( f(x) \) is NOT continuous at \( x = -2 \).

(b) Note that one-side is not necessary, but for completeness, it is shown.
\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3 + 2x^2 - 8x}{x^2 - 4}
= \lim_{x \to 2} \frac{x(x + 4)(x - 2)}{(x + 2)(x - 2)} = \lim_{x \to 2} \frac{x(x + 4)}{x + 2}
= \frac{2(2 + 4)}{2 + 2} = 3
\]

\[
\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x(x + 4)}{x + 2} = \frac{2(2 + 4)}{2 + 2} = 3
\]

Yes, the limit exists, that is, \( \lim_{x \to 2} f(x) = 3 \).

(c) Note that \( f(2) = 4 \neq 3 = \lim_{x \to 2} f(x) \)

By the definition of continuity, \( f(x) \) is NOT continuous at \( x = 2 \).