Area between Curves

The area of the region between the curves \( y = f(x) \) and \( y = g(x) \) and between \( x = a \) and \( x = b \) is

\[
A = \int_a^b |f(x) - g(x)| \, dx.
\]

Keep in mind that the term “region between the curves” refers to a bounded region (i.e., to a region that does not extend to infinity).

Instead of the absolute value, if the graphs of the two functions bounding the region do not intersect within \((a, b)\), then

\[
A = \int_{x=a}^{x=b} y(x)_{\text{upper}} - y(x)_{\text{lower}} \, dx
\]

Example. Find the area of the region bounded by the curves \( y = e^x \), \( y = 5^x \), and \( x = 1 \).

Solution. Sketch the curves and identify the region. Note that the curves intersect at \( x = 0 \).
On \([0, 1]\), the upper curve is \(y = 5^x\), and the lower curve is \(y = e^x\).

The area of the region is:

\[
A = \int_0^1 (5^x - e^x) \, dx = \left[ \frac{5^x}{\ln 5} - e^x \right]_0^1 \\
= \left( \frac{5}{\ln 5} - e^1 \right) - \left( \frac{5^0}{\ln 5} - e^0 \right) = \frac{5}{\ln 5} - e - \left( \frac{1}{\ln 5} \right) + 1 \\
= \frac{4}{\ln 5} - e + 1
\]

If the region involved looks simpler when the bounding curves are viewed as functions of \(y\), then the analogous formula is used to integrate with respect to \(y\):

\[
A = \int_{y=a}^{y=b} x(y)_{right} - x(y)_{left} \, dy
\]

Example. Find the area of the region enclosed by the curves \(y = x^3\) and \(y = x^2\).

Solution. Sketch the area of the region.
To find the points where the curves intersect, combine the two equations:

\[ x^3 = x^2 \]
\[ x^3 - x^2 = 0 \]
\[ x^2 (x - 1) = 0 \]

Thus, \( x = 0 \) and \( x = 1 \). The corresponding points are \((0, 0)\) and \((1, 1)\).

The right curve is \( y = x^3 \), or \( x = y^{1/3} \), and the right curve is \( y = x^2 \), or \( x = \sqrt{y} \).

The area of the region is

\[
A = \int_0^1 (y^{1/3} - y^{1/2}) \, dy = \left. \left( \frac{3}{4} y^{4/3} - \frac{2}{3} y^{3/2} \right) \right|_0^1
\]

\[
= \left( \frac{3}{4} (1)^{4/3} - \frac{2}{3} (1)^{3/2} \right) - \left( \frac{3}{4} (0)^{4/3} - \frac{2}{3} (0)^{3/2} \right) = \frac{1}{12}
\]