Proof by Contrapositive:

Proof by contrapositive can be used to prove if-then statements indirectly. The statement $P \Rightarrow Q$ is logically equivalent to its contrapositive $\neg Q \Rightarrow \neg P$. Since *logically equivalent* means that when one is true the other must also be true, if we prove that the contrapositive is true, we also proved the original statement. Hint: use the contrapositive when there are “negative” statements (no solutions, not natural, $\neq$, $\not\in$, etc.), when the if part is hard to start from, or simply when stuck.

**Example.** Use the contrapositive the prove that if $c > \frac{49}{8}$ then $f(x) = 2x^2 + 7x + c$ has no solutions.

**Solution.**
First, we formulate the contrapositive:

- $P$ is $c > \frac{49}{8}$ so $\neg P$ is $c \leq \frac{49}{8}$.
- $Q$ is $f(x) = 2x^2 + 7x + c$ has no solutions so $\neg Q$ is $f(x) = 2x^2 + 7x + c$ has at least one solution.

Therefore, the contrapositive is if $f(x) = 2x^2 + 7x + c$ has at least one solution then $c \leq \frac{49}{8}$. For a quadratic polynomial $f(x) = ax^2 + bx + c$ to have roots its
determinant must satisfy $b^2 - 4ac \geq 0$. For our polynomial that is $(7)^2 - 4(2)(c) \geq 0$. We try to isolate $c$:

$$
(7)^2 - 4(2)(c) \geq 0 \\
49 - 8c \geq 0 \\
49 \geq 8c \\
\frac{49}{8} \geq c
$$

which was what we wanted. Since the contrapositive is equivalent to the original statement, we have proven that if $c > \frac{49}{8}$ then $f(x) = 2x^2 + 7x + c$ has no solutions.

**Proof by Contradiction:**

Proof by contradiction can be used to prove any statement, including if-then. To prove a statement $P$ by contradiction:

1. Formulate $\neg P$ and assume it is true.
2. Manipulate the statement(s) in $\neg P$ to find a contradiction. A contradiction means two opposing statements being true at the same time (e.g. $x = 0$ and $x \neq 0$).

Hint: use contradiction when there are negative statements, “or” statements or simply when stuck.

**Example.** Prove that if $a^2 + 5$ is even then $a$ is odd.

**Solution.**

1. The statement to prove in this case has the form $P \Rightarrow Q$, where $P$ means $a^2 + 5$ is even and $Q$ means $a$ is odd. The negation of
2. \( P \Rightarrow Q \) is \( P \wedge \neg Q \), that would be \( a^2 + 5 \) is even and \( a \) is even.

3. Now, we must assume these facts are true and use them to achieve a contradiction.

We’ve assumed that \( a \) is even, that means \( a = 2k \) for some \( k \in \mathbb{Z} \). We can use this information about \( a \) and plug it into \( a^2 + 5 \):

\[
\begin{align*}
    a^2 + 5 &= (2k)^2 + 5 \\
    &= 4k^2 + 5 \\
    &= 4k^2 + 4 + 1 \\
    &= 2(2k^2 + 2) + 1
\end{align*}
\]

Since \( a^2 + 5 \) takes the form \( 2m + 1 \) for \( m \in \mathbb{Z} \), it must be odd. However, we originally said \( a^2 + 5 \) is even. This is a contradiction; therefore the negation must be false and the original statement must be true. \( \blacksquare \)