University of Toronto Mississauga

Mathematical and Computational Sciences

MAT157Y5Y- Term Test 1

Duration - 110 minutes

No Aids Permitted

Surname/Family Name: ______________________________

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This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in. No aids are permitted on this examination. Examples of illegal aids include, but are not limited to, textbooks, notes, calculators, or any electronic device.

Your solutions to each problem should be written in the provided space only. Scratch paper is available upon request; however, scratch work cannot be submitted for marks. **Do not, under any circumstances, write on the QR code in the top right corner of each page. Doing so will result in you receiving a zero (0) on this test.**
Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work, will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

1. (10 points) A function $f : B \to C$ is said to be a *monomorphism* if whenever $g_1, g_2 : A \to B$ are a pair of functions such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$. Show that f is a monomorphism if and only if f is injective.

Solution: Suppose that f is a monomorphism, and that $b_1, b_2 \in B$ satisfy $f(b_1) = f(b_2)$. Let $g_1 : B \to B$ be the constant function $g_1(x) = b_1$ for all $x \in B$, and similarly $g_2 : B \to B$ is such that $g_2(x) = b_2$. Now $f(g_1(x)) = f(b_1) = f(b_2) = f(g_2(x))$ for all $x \in B$, so by assumption $g_1(x) = g_2(x)$ for all $x \in B$, which in turn shows that $b_1 = b_2$.

Conversely, suppose that f is injective. By a result from class we know that f is left invertible, so choose one such inverse $h : C \to B$ such that $h \circ f = \mathrm{id}_B$. Let $g_1, g_2 : A \to B$ be any to functions such that $f \circ g_1 = f \circ g_2$, and post-compose by h to get

$$\begin{aligned} h \circ (f \circ g_1) &= (h \circ f) \circ g_1 = \mathrm{id}_B \circ g_1 = g_1 \\ &= h \circ (f \circ g_2) && \text{by assumption} \\ &= (h \circ f) \circ g_2 = \mathrm{id}_B \circ g_2 = g_2. \end{aligned}$$

This shows that $g_1 = g_2$, allowing us to conclude that f is a monomorphism as required.

2. (10 points) A set $S \subseteq \mathbb{R}$ is said to be *open* if for every $s \in S$, there exists a positive real number r such that the set $B(s,r) = \{y \in \mathbb{R} : |s - y| < r\}$ is contained within S; namely, $B(s,r) \subseteq S$. Show that the set $S = (0,1)$ is open.

Solution: Fix some $x \in (0,1)$. The radius we need to use is the smaller of the distances from either 0 or 1, so let $r = \min\{x, 1-x\}$, both of which are positive by the assumption that $0 < x < 1$. We claim that $B(x,r) \subseteq S$. Indeed, let $y \in B(x,r)$ so that $|y - x| < r$ by assumption. Thus

$$|y| = |y - x + x| \leq |y - x| + |x| \leq r + x < (1 - x) + x = 1.$$

Hence $y \in (-1,1)$. Moreover, we can write $|y - x| < r$ as $x - r < y < x + r$. Focusing on the first half of the inequality and using the fact that $r \leq x$ implies that $x - r \geq 0$, so $0 \leq x - r < y$ shows that $y > 0$. Both the inequalities combined shows that $y \in (0,1)$.

3. (10 points) Let $D = \left\{x \in \mathbb{Q} : x = \dfrac{m}{2^n}, m \in \mathbb{Z}, n \in \mathbb{N}\right\}$. Show that D is dense in $\mathbb{R}$; namely, that every open interval (a, b) contains at least one element of D. *Hint:* Choose n sufficiently large so that $1/2^n < b - a$, and argue that some point of the form $m/2^n$ ($m \in \mathbb{Z}$) must live in (a, b).

Solution: Fix an open interval (a, b) for some $a, b \in \mathbb{R}$ with $a < b$. Per the suggestion, choose $n \in \mathbb{N}$ sufficiently large so that $2^{-n} < b - a$. Define the set $B = \{m/2^n : m \in \mathbb{Z}\}$, of which we claim that one of the elements in B is also in (a, b). For the sake of contradiction, assume there is no such value in B which satisfies this definition. Let M be the largest such integer satisfying $M/2^n < a$, so that $(M + 1)/2^n > b$. But then

$$b - a < \frac{M+1}{2^n} - \frac{M}{2^n} = \frac{1}{2^n} < b - a$$

a contradiction, thus some element of B must live in (a, b).

4. (10 points) Suppose $f : (-1, 1) \to \mathbb{R}$. Using the ϵ-δ definition of the limit, show that

$$\lim_{x\to 0} f(x) = 0 \quad \text{if and only if} \quad \lim_{x\to 0} |f(x)| = 0.$$

Note: This can be done directly. If you wish to employ a theorem from class to do this, you must first prove that theorem.

Solution: The crux of this entire result hinges upon the fact that $||x|| = |x|$. Indeed, we know that $|x| \geq 0$ for all x, and that $|x| = x$ if $x \geq 0$, so $||x|| = |x|$. We'll use this result freely in the following arguments.

Suppose that $\lim_{x\to 0} f(x) = 0$, for which we want to show that the absolute value similarly goes to 0. Let $\epsilon > 0$ be given, and choose a $\delta > 0$ such that $|f(x) - 0| = |f(x)| < \epsilon$ whenever $0 < |x - 0| = |x| < \delta$. We claim this same δ works for the absolute value of f. Indeed, if $0 < |x| < \delta$, then

$$||f(x)| - 0| = ||f(x)|| = |f(x)| < \epsilon.$$

The converse direction is almost identical. Suppose $|f(x)| \xrightarrow{x\to 0} 0$. Let $\epsilon > 0$ be given and choose $\delta > 0$ such that $||f(x)| - 0| = |f(x)| < \epsilon$. Clearly setting $\delta = \epsilon$ will ensure the desired inequality.

5. (10 points) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ satisfy

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M$$

for some $c, L, M \in \mathbb{R}$. Using the ϵ-δ definition of the limit, show that

$$\lim_{x \to c} [2f(x) - 3g(x)] = 2L - 3M.$$

Note: This can be done directly. If you wish to employ a theorem from class to do this, you must first prove that theorem.

Solution: Let $\epsilon > 0$ be given, and fix $\delta_f, \delta_g > 0$ such that

$$\begin{aligned} 0 < |x - c| < \delta_f \quad &\Rightarrow \quad |f(x) - L| < \frac{\epsilon}{4} \\ 0 < |x - c| < \delta_g \quad &\Rightarrow \quad |g(x) - M| < \frac{\epsilon}{6}. \end{aligned}$$

Let $\delta = \min\{\delta_f, \delta_g\}$ and suppose that $0 < |x - c| < \delta$, so that both inequalities above will be true. Thus

$$\begin{aligned} |[2f(x) - 3g(x)] - [2L - 3M]| &= |2(f(x) - L) - 3(g(x) - M)| \leq 2|f(x) - L| + 3|g(x) - M| \\ &< 2\frac{\epsilon}{4} + 3\frac{\epsilon}{6} = \epsilon, \end{aligned}$$

as required.

6. (i) (3 points) Suppose $f : \mathbb{R} \to \mathbb{R}$. Without using the word "not", or any synonym of "not", find the mathematical statement for $\lim_{x \to c} f(x) \neq L$.

Solution: The definition of the limit is $\forall \epsilon > 0, \exists \delta > 0, \forall x, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$. Negating gives

$$\exists \epsilon > 0, \forall \delta > 0, \exists x, 0 < |x - c| < \delta \text{ and } |f(x) - L| \geq \epsilon.$$

(ii) (7 points) Show that $\lim_{x \to 0} \frac{x}{|x|} \neq 1$.

Solution: Let $\epsilon = 1$, let $\delta > 0$ be arbitrarily given, and choose $x = -\delta/2$. Note that

$$\left| \frac{x}{|x|} - 1 \right| = \left| \frac{-\delta/2}{|-\delta/2|} - 1 \right| = |-1 - 1| = 2 > \epsilon$$

showing that the limit cannot be 1, as required.