## TheRobert Gillespie ACADEMIC SKILLS CENTRE

## Logical Equivalence and Negating Logical Statements

Two logical statements are equivalent if the values in their truth tables are all equal.

Example. Show $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$.

## Solution.

To make the truth table for logical statement(s) we break them down into their constituent parts. For example $\neg(P \Rightarrow Q)$ is made up from $P$ and $Q$, which are put together as $P \Rightarrow Q$ and finally negated into $\neg(P \Rightarrow Q)$.

In $P \wedge \neg Q$ we still have $P$ and $Q$, but we also need $\neg Q$ which together with $P$ makes $P \wedge \neg Q$.

First we fill in the columns for the statements $P$ and $Q$, making sure we have every combination of true and false (i.e., every combination of their values).

| $P$ | $Q$ | $\neg Q$ | $P \wedge \neg Q$ | $P \Rightarrow Q$ | $\neg(P \Rightarrow Q)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| T | T |  |  |  |  |
| T | F |  |  |  |  |
| F | T |  |  |  |  |
| F | F |  |  |  |  |

Then we fill in the rest of the columns using the tables for the other connectives (not, and, or, ...).

[^0]| $P$ | $Q$ | $\neg Q$ | $P \wedge \neg Q$ | $P \Rightarrow Q$ | $\neg(P \Rightarrow Q)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| T | T | F | F | T | F |
| T | F | T | T | F | T |
| F | T | F | F | T | F |
| F | F | T | F | T | F |

Since the columns for $\neg(P \Rightarrow Q)$ and $P \wedge \neg Q$ are the same, the statements are logically equivalent.

## Negating Statements

We negate a statement by using the following formulas/rules:

$$
\begin{array}{ll}
\text { 1. } & \neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q \\
\text { 2. } & \neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q \\
\text { 3. } & \neg(P \Rightarrow Q) \Leftrightarrow P \wedge \neg Q \\
\text { 4. } & \neg((\forall x) R(x)) \Leftrightarrow(\exists x) \neg R(x) \\
\text { 5. } & \neg((\exists x) R(x)) \Leftrightarrow(\forall x) \neg R(x)
\end{array}
$$

Note that in negation, we switch and $(\wedge)$ and or $(\vee)$ and for every $(\forall)$ and there exists ( $\exists$ ).

Example. Show the negation of

$$
(\forall x \in F)[x \neq 0 \Rightarrow(\exists y \in F)(y \neq 0 \wedge x \cdot y=1)]
$$

and simplify so that the formula does not include the negation symbol $\neg$.

## Solution.

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$$
\begin{aligned}
& \neg\{(\forall x \in F)[x \neq 0 \Rightarrow((\exists y \in F)(y \neq 0 \wedge x \cdot y=1))]\} \\
& (\exists x \in F) \neg[x \neq 0 \Rightarrow((\exists y \in F)(y \neq 0 \wedge x \cdot y=1))] \\
& (\exists x \in F)[x \neq 0 \wedge \neg((\exists y \in F)(y \neq 0 \wedge x \cdot y=1))] \\
& (\exists x \in F)[x \neq 0 \wedge((\forall y \in F) \neg(y \neq 0 \wedge x \cdot y=1))] \\
& (\exists x \in F)[x \neq 0 \wedge((\forall y \in F)(y=0 \vee x \cdot y \neq 1))]
\end{aligned}
$$

Example. Compute the negative of $(\exists x \in \mathbb{R})\left(\left((\forall y \in \mathbb{R})\left(y^{2} \neq x\right)\right) \vee(x \geq 0)\right)$ without using the $\neg$ symbol.

## Solution.

$$
\begin{aligned}
& \neg\left\{(\exists x \in \mathbb{R})\left(\left((\forall y \in \mathbb{R})\left(y^{2} \neq x\right)\right) \vee(x \geq 0)\right)\right\} \\
& (\forall x \in \mathbb{R}) \neg\left(\left((\forall y \in \mathbb{R})\left(y^{2} \neq x\right)\right) \vee(x \geq 0)\right) \\
& (\forall x \in \mathbb{R})\left(\neg\left((\forall y \in \mathbb{R})\left(y^{2} \neq x\right)\right) \wedge \neg(x \geq 0)\right) \\
& (\forall x \in \mathbb{R}))\left(\left((\exists y \in \mathbb{R}) \neg\left(y^{2} \neq x\right)\right) \wedge(x<0)\right) \\
& (\forall x \in \mathbb{R})\left(\left((\exists y \in \mathbb{R})\left(y^{2}=x\right)\right) \wedge(x<0)\right)
\end{aligned}
$$


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