AGM Inequality

For real numbers A and B the Arithmetic-Geometric Mean Inequality says:

- $A \cdot B \le \left(\dfrac{A+B}{2}\right)^2$.
- If $A, B \ge 0$ then $\sqrt{AB} \le \dfrac{A+B}{2}$.

The quantity $\sqrt{AB}$ is called the geometric mean, and $\dfrac{A+B}{2}$ is the arithmetic mean of A and B. These inequalities become equalities if and only if $A = B$.

We can use the AGM inequality to **find maximum and minimum** values by substituting for A and B above. Keep in mind that we have multiplication on the "smaller" side and addition on the "greater" side**.** Thus, we can find a **maximum** for the multiplication side ($A \cdot B \le M$) if the addition side can be dealt with and a **minimum** for the addition side ($M \le \left(\dfrac{A+B}{2}\right)^2$) if the multiplication side can be dealt with.

Example. Find the maximum area of a rectangle with a perimeter of $20m$.
What dimensions give the maximum area?

Solution.

We would like to find a maximum for the area. The area of a rectangle is given by $A = l \cdot w$, a maximum for the area would then mean finding a number M so that $l \cdot w \le M$.

We're told the perimeter is $20m$, this is given by $2l+2w=20$, i.e., $l+w=10$. We use AGM by setting $A=l$ and $B=w$ which gives:

This is useful because the left side contains $l \cdot w$ and on the right side we use $l+w=10$ to look for our M.

$$(l)(w) \le \left(\frac{l+w}{2}\right)^2$$
$$l \cdot w \le \left(\frac{10}{2}\right)^2$$
$$l \cdot w \le 25$$

Therefore, we see that the maximum area is $25m^2$.

We want to know when $l \cdot w = 25$. AGM says this happens when $A=B$, that is $l=w$, which we can use with $l+w=10$:

$$l+l=10$$
$$l=5$$

Substituting $l=5$ into $l=w$ tells us that $w=5$, meaning the rectangle with the maximum area is a 5×5 square. $\square$

Example. Find a lower bound for $f(x)=x+\frac{9}{x}+1$ when $x>0$. Find the point(s) where this happens.

Solution.
We're being asked to find a lower bound, or a minimum, so we'll have to use an inequality. The key thing to notice is that we have $x+\frac{9}{x}$, if we multiply x and $\frac{9}{x}$ we can eliminate the x; this will allow us to use AGM. Since we have $x>0$ we can use the second version with $A=x$ and $B=\frac{9}{x}$:

$$
\begin{aligned}
\sqrt{AB} &\le \frac{A+B}{2} \\
\sqrt{x\frac{9}{x}} &\le \frac{x+\frac{9}{x}}{2} \\
2\sqrt{9} &\le x+\frac{9}{x} \\
6 &\le x+\frac{9}{x}
\end{aligned}
$$

At this point we almost have the function. We can add 1 to both sides to complete it:

$$
\begin{aligned}
7 &\le x+\frac{9}{x}+1 \\
7 &\le f(x)
\end{aligned}
$$

We want to know x where $f(x)=7$, according to AGM this happens when $A=B$:

$$
\begin{aligned}
x &= \frac{9}{x} \\
x^2 &= 9 \\
x &= \pm 3
\end{aligned}
$$

Since we know that $x>0$ the solution is $x=3$. Thus, the point is $(3,7)$. □