The Robert Gillespie ACADEMIC SKILLS CENTRE

The Fundamental Theorem of Calculus

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Suppose f is continuous on $[a,b]$.

Part 1. If $g(x)=\int_a^x f(t)\,dt$, then $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b) and $g'(x)=f(x)$.

Part 2. $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$, where F is any antiderivative of f, that is, $F'=f$.

Example. If $g(x)=\int_5^{-x^3} e^{3t^2+1}dt$, find $g'(x)$.

Solution. Let $u=-x^3$. Then, $\dfrac{du}{dx}=-3x^2$.

By Chain Rule, compute $g'(x)$:

$$
\begin{aligned}
g'(x) &= \frac{d(g(x))}{dx} = \frac{d}{dx}\left(\int_5^{-x^3} e^{3t^2+1}dt\right) \\
&= \frac{d}{dx}\left(\int_5^{u} e^{3t^2+1}dt\right) = \frac{d}{du}\left(\int_5^{u} e^{3t^2+1}dt\right)\frac{du}{dx} \\
&= (e^{3u^2+1})(-3x^2) \\
&= -3x^2 e^{3(-x^3)^2+1} = -3x^2 e^{3x^6+1}
\end{aligned}
$$

Example. Evaluate each integral.

(a) $\int_1^2 \left(x^2 + \frac{1}{x} \right) dx$ (b) $\int_2^4 \left(x^{1/2} + e^{4x} \right) dx$

Solution. (a)

$$\int_1^2 \left(x^2 + \frac{1}{x} \right) dx = \left(\frac{x^3}{3} + \ln x \right) \Bigg|_1^2$$

$$= \left(\frac{2^3}{3} + \ln 2 \right) - \left(\frac{1^3}{3} + \ln 1 \right)$$

$$= \frac{7}{3} + \ln 2$$

(b)

$$\int_4^9 \left(x^{1/2} + e^{4x} \right) dx = \left(\frac{x^{3/2}}{\frac{3}{2}} + \frac{e^{4x}}{4} \right) \Bigg|_4^9$$

$$= \left(\frac{2x^{3/2}}{3} + \frac{e^{4x}}{4} \right) \Bigg|_4^9$$

$$= \left(\frac{2(9)^{3/2}}{3} + \frac{e^{36}}{4} \right) - \left(\frac{2(4)^{3/2}}{3} + \frac{e^{16}}{4} \right)$$

$$= \frac{54}{3} + \frac{e^{36}}{4} - \frac{16}{3} - \frac{e^{16}}{4}$$

$$= \frac{38}{3} + \frac{e^{36}}{4} - \frac{e^{16}}{4}$$