The Robert Gillespie ACADEMIC SKILLS CENTRE

Power Series

Power Series

A power series (centred at zero) is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$ where x is a variable and the c_n's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$ is called a power series in $(x-a)$ or a power series centred at a.

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

(1) The series converges only when $x = a$.
(2) The series converges for all x.
(3) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (3) is called the radius of convergence of the power series. In case (1), it is said that $R = 0$, and in case (2), $R = \infty$.

The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. In case (1), the interval consists only one point a; in case (2), the interval is $(-\infty, \infty)$; and, in case (3), the inequality $|x-a| < R$ can be written as $a - R < x < a + R$. Considering the endpoints $x = a \pm R$, there are four possibilities: $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$, or $[a-R, a+R]$. The endpoints have to be checked individually for convergence.

Example. Find the radius of convergence and the interval of convergence for the power series $\sum_{n=1}^{\infty}(-1)^n \frac{(x-2)^n}{n+1}$.

Solution. By the Ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{n+2}}{\frac{(x-2)^n}{n+1}}\right| = \lim_{n\to\infty}\left|\left(\frac{(x-2)^{n+1}}{n+2}\right)\left(\frac{n+1}{(x-2)^n}\right)\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n+2}(x-2)\right| = \lim_{n\to\infty}\left|\frac{1+1/n}{1+2/n}(x-2)\right|$$

$$= \left|\frac{1+0}{1+0}(x-2)\right| = |x-2| < 1$$

Thus, the series is absolutely convergent for all x values which satsify $|x-2|<1$.

The radius of convergence is $R=1$.

Using the properties of absolute value, rewrite $|x-2|<1$as

Using absolute value properties,

$$|x-2|<1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

To find the interval of convergence, examine the endpoints.

At the endpoints of the inequality, the series may be conditionally convergent or divergent. Substitute each endpoint in the series, and test for converges.

When $x=1$,

$$\sum_{n=1}^{\infty}(-1)^n\frac{(1-2)^n}{n+1}=\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n+1}=\sum_{n=1}^{\infty}\frac{1}{n+1}.$$

Note that $\sum_{n=1}^{\infty}(-1)^{2n}=\sum_{n=1}^{\infty}1=1+1+1+1+1+\ldots$

This is a p-series (starting with the value $1/2$) with $p=1$, and so it diverges.

When $x=3$, $\sum_{n=1}^{\infty}(-1)^n\frac{(3-2)^n}{n+1}=\sum_{n=1}^{\infty}(-1)^n\frac{(1)^n}{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$.

By the Alternating Series Test, this series converges.

Thus, the interval of convergence for the given power series is $(1,3]$.