The Robert Gillespie ACADEMIC SKILLS CENTRE

One-sided Limits

One-Sided Limits

If $f(x)$ approaches a real number L as x approaches a from the left $(x < a)$, then

$$\lim_{x\to a^-} f(x) = L \qquad \text{(left-handed one sided limit)}$$

Likewise, if $f(x)$ approaches a real number M as x approaches a from the right $(x > a)$, then

$$\lim_{x\to a^+} f(x) = M \qquad \text{(right-handed one sided limit)}$$

Existence of a Limit of a Function

Theorem A function $f(x)$ has a limit as approaches a number a if and only if its left-handed and right-handed limits that exist and are equal:

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Example. For the function $f(x) = \begin{cases} x^3 - 3x, & \text{if } -1 \le x < 1 \\ x - 5, & \text{if } x \ge 1 \end{cases}$, evaluate $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$. Does the limit $\lim_{x\to 1} f(x)$ exist?

Solution. Since $f(x) = x - 5$ for $x \ge 1$,

$$\lim_{x\to 1^+}(x-5) = 1 - 5 = -4$$

Since $f(x) = x^3 - 3x$ for $-1 \le x < 1$,

$$\lim_{x\to 1^-} x^3 - 3x = \lim_{x\to 1^-}(1)^3 - 3(1) = 1 - 3 = -2$$

Since $\lim_{x\to 1^-} f(x) = -2 \ne -4 = \lim_{x\to 1^+} f(x)$,

$\lim_{x \to 1} f(x)$ does not exist.

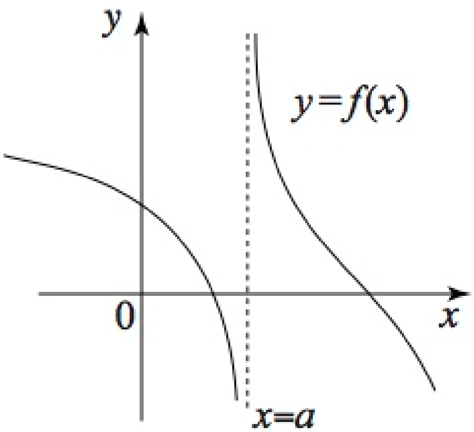
Infinite Limits

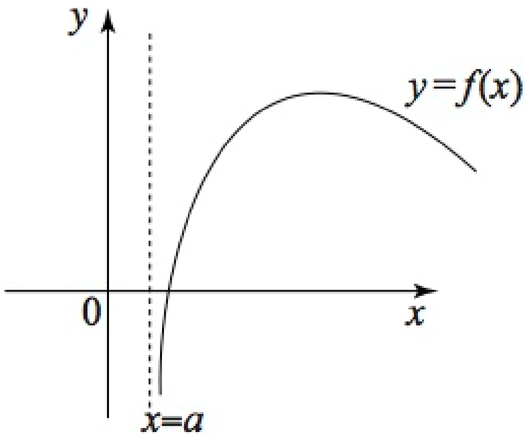
If $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$, then it is said that $\lim_{x \to a} f(x)$ does not exist.

Vertical Asymptotes

The line $x = a$ is a vertical asymptote of the graph of $y = f(x)$ if either $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$.

Geometrically, the graph of a function $f(x)$ is said to have a vertical asymptote at $x = a$ if $f(x)$ increases or decreases without bound as x approaches a, from either the right or the left, or from both directions.





In general, a rational function $R(x) = \dfrac{p(x)}{q(x)}$ has a vertical asymptote $x = a$ whenever $q(a) = 0$, but $p(a) \neq 0$.

Example. Find $\lim_{x\to 3^+} \frac{2}{x-3}$ and $\lim_{x\to 3^-} \frac{2}{x-3}$. Is there a vertical asymptote?

Solution. When $x=3$, the denominator in $f(x)=\frac{2}{x-3}$ is 0 and the numerator is a nonzero number. Thus, the limit of $f(x)$ as x approaches 3 is either ∞ or $-\infty$.

Consider the right-handed limit $\lim_{x\to 3^+} \frac{2}{x-3}$.

Since $x>3$, the denominator $x-3$ is positive, and thus, the fraction $\frac{2}{x-3}$ is positive.

Therefore, $\lim_{x\to 3^+} \frac{2}{x-3}=+\infty$, and the function $f(x)=\frac{2}{x-3}$ has a vertical asymptote at $x=3$.

Compute the left-handed limit $\lim_{x\to 3^-} \frac{2}{x-3}$ in the same way. Alternatively, pick a value of x very close to the left of 3, say $x=2.9$, and substitute this value into $\frac{2}{x-3}$. A negative number is obtained. It follows that $\lim_{x\to 3^-} \frac{2}{x-3}=-\infty$. Thus, the function $f(x)=\frac{2}{x-3}$ has a vertical asymptote at $x=3$.

As a result, the function $f(x)=\frac{2}{x-3}$ has a vertical asymptote at $x=3$ from both sides.