The Robert Gillespie ACADEMIC SKILLS CENTRE

Definition of a Derivative

Derivative of a Function $f(x)$ at a Number a

$f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, or $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, provided that the limit exists. (Note that the former formula is used more often than the latter.)

Example. Using the definition of the derivative at a number, find the slope of tangent line at $x = -2$ to the curve $y = -6x^2$. What is the equation of the tangent line at this point?

Solution. By definition of the derivative, the slope of the tangent line at $x = -2$ is

$$\begin{aligned} f'(-2) &= \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h} \\ &= \lim_{h \to 0} \frac{-6(-2+h)^2 - (-6(-2)^2)}{h} \\ &= \lim_{h \to 0} \frac{-24 + 24h - 6h^2 + 24}{h} \\ &= \lim_{h \to 0} \frac{h(24 - 6h)}{h} = \lim_{h \to 0}(24 - 6h) = 24 \end{aligned}$$

When $x = -2$, $y(-2) = -6(-2)^2 = -24$, that is, the point of tangency is $(-2, -24)$.

Using the point-slope formula, $\begin{aligned} y + 24 &= 24(x+2) \\ y &= 24x + 24 \end{aligned}$.

Derivative Function $f'(x)$ of a Function $f(x)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$, for any x for which this limit exists.

Example. Using the definition of the derivative, given the function $f(x) = 3 - \dfrac{1}{x^2}$, find $f'(x)$.

Solution. By the definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\left[3-\dfrac{1}{(x+h)^2}\right]-\left[3-\dfrac{1}{x^2}\right]}{h} = \lim_{h\to 0}\frac{-\dfrac{1}{(x+h)^2}+\dfrac{1}{x^2}}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{-x^2+(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0}\left(\frac{-x^2+x^2+2xh+h^2}{x^2(x+h)^2}\right)\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{hx^2(x+h)^2} = \lim_{h\to 0}\frac{2x+h}{x^2(x+h)^2} = \frac{2x}{x^2(x)^2} = \frac{2}{x^3}$$