

Absolute Convergence & Conditional Convergence

Absolutely Convergent Series

Given a series
$$\sum_{n=1}^{\infty} a_n$$
, consider the corresponding series
 $\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \dots$ The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem: If a series is absolutely convergent, then it is convergent.

Note that the reserve is not true (the series in Example (a) below is convergent, but not absolutely convergent).

Conditionally Convergent Series

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if it is convergent, but not absolutely convergent.

Example. Determine if each series is absolutely convergent or conditionally convergent.

(a)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$
 (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}$

Solution.

(a)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}.$$
 This is a p-series with $p = 1$, and thus, diverges.

Creative This work is licensed under the Creative Commons Attribution NonCommercial-ShareAlike 4.0 International License. To view a copy of this license, visit <u>http://creativecommons.org/licenses/by-nc-sa/4.0</u>. But, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent. (b) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{3^n} \right| = \sum_{n=1}^{\infty} \frac{1}{3^n}$. This is a geometric series with $r = \frac{1}{3} < 1$ which converges. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n}$ is absolutely convergent (and thus, convergent as well).

