The Robert Gillespie ACADEMIC SKILLS CENTRE

Relative Maximum and Relative Minimum

Relative Maximum and Relative Minimum

A function $f(x,y)$ is said to have a relative maximum at the point $P(a,b)$ in the domain of f if $f(a,b) \geq f(x,y)$ for all points (x,y) in a disk centered at P. Similarly, if $f(c,d) \leq f(x,y)$ for all points (x,y) in a circular disk centered at Q, then $f(x,y)$ has a relative minimum at $Q(c,d)$.

Critical Points

A point (a,b) in the domain of $f(x,y)$ is called a critical point of f, if

$$f_x(a,b) = 0 \qquad \text{and} \qquad f_y(a,b) = 0,$$

or if one of these partial derivative does not exist.

Theorem: If f has a relative extreme value at (a,b), and the partial derivatives f_x and f_y exist, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

This theorem says that the relative extrema of f can occur only at critical points.

Saddle Points

Although all relative extrema of a function must occur at critical points, not every critical point yields a relative extremum. A critical point which is neither a local minimum nor a local maximum is called a saddle point.

A saddle point a function indeed looks like a saddle: it has a relative maximum in some direction and a relative minimum in another direction (in simplest cases, such as $f(x,y) = y^2 - x^2$, these directions are the directions of the x-axis and the y-axis).

Finding Relatiive Extrema Using the Second Derivative Test

The procedure involving second-order derivatives can be used to decide whether a given critical point is a relative maximum, a relative minimum, or a saddle point.

Let $f(x,y)$ be a function of two variables x and y whose partial derivatives f_x, f_y, f_{xx}, f_{yy}, and f_{xy} all exist and are continuous on a disk centred at each critical point, and let

$$D(x,y) = f_{xx}(x,y) f_{yy}(x,y) - [f_{xy}(x,y)]^2 .$$

Step 1. Find all critical points of $f(x,y)$, that is, all points (a,b) so that

$$f_x(a,b) = 0 \quad \text{and} \quad f_y(a,b) = 0 .$$

Step 2. For each critical point (a,b) found in Step 1, evaluate $D(a,b)$.

Step 3. If $D(a,b) < 0$, then (a,b) is a saddle point (and the function has no extreme value there).

If $D(a,b) > 0$, compute $f_{xx}(a,b)$:

If $f_{xx}(a,b) > 0$, then (a,b) is a relative minimum.
If $f_{xx}(a,b) < 0$, then (a,b) is a relative maximum.
If $D(a,b) = 0$, the test is inconclusive and f may have either a relative extremum or a saddle point at (a,b).

In summary:

Sign of D	Sign of f_{xx}	Behavior at (a,b)
+	+	relative minimum
+	-	relative maximum
-	not relevant	saddle point

Example. Find all critical points for the function $f(x,y) = 12x - x^3 - 4y^2$ and classify each as a relative maximum, a relative minimum, or a saddle point.

Solution. Solving $f_x = 12 - 3x^2$ and $f_y = -8y$ yields $x = 2$, $x = -2$, and $y = 0$. Thus, the critical points are $(2, 0)$ and $(-2, 0)$.
The second-order partial derivatives are $f_{xx} = -6x$, and $f_{xy} = 0$.

Then, $D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (-6x)(-8) - 0 = 48x$.

At $(2, 0)$: $D(2, 0) = 48(2) = 96 > 0$ and $f_{xx}(2, 0) = -6(2) = -12 < 0$ which means that $(2, 0)$ is a relative maximum.

At $(-2, 0)$: $D(-2, 0) = 48(-2) = -96 < 0$ which means that $(-2, 0)$ is a saddle point.

Example. Find all critical points for the function $f(x, y) = e^{2xy}$ and classify each as a relative maximum, a relative minimum, or a saddle point.

Solution. Since $f_x = 2ye^{2xy}$ and $f_y = 2xe^{2xy}$, there is only one critical point $(0, 0)$.

The second-order partial derivatives are $f_{xx} = 4y^2e^{2xy}$, $f_{yy} = 4x^2e^{2xy}$, and $f_{xy} = 2e^{2xy} + 4xye^{2xy}$.

Then,

$$D(x, y) = f_{xx} f_{yy} - (f_{xy})^2$$
$$= (4y^2e^{2xy})(4x^2e^{2xy}) - (2e^{2xy} + 4xye^{2xy})^2$$

At $(0, 0)$, $D(0, 0) = 0 - (2e^0 + 0)^2 = -4 < 0$, which means that $(0, 0)$ is a saddle point.