Lagrange Multipliers

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The method of Lagrange multipliers uses the fact that any relative extremum of the function $f(x,y)$ subject to the constraint $g(x,y)=k$, if it exists, must occur at a critical point (a,b) of the function

$$F(x,y)=f(x,y)-\lambda[g(x,y)-k]$$

where λ is a new variable (called the Lagrange multiplier). To find the critical points of F, compute its partial derivatives:

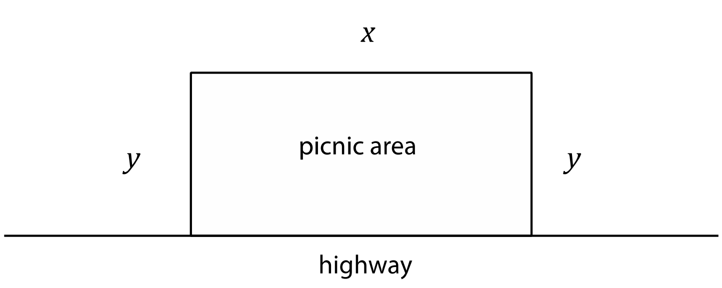
$$F_x=f_x-\lambda g_x\ , \quad F_y=f_y-\lambda g_y\ , \quad F_\lambda=-(g-k)\ ,$$

and solve the equations $F_x=0$, $F_y=0$, and $F_\lambda=0$ simultaneously, as follows, to obtain three (Lagrange) equations:

$$F_x=f_x-\lambda g_x=0 \qquad \text{or} \qquad f_x=\lambda g_x$$
$$F_y=f_y-\lambda g_y=0 \qquad \text{or} \qquad f_y=\lambda g_y$$
$$F_\lambda=-(g-k)=0 \qquad \text{or} \qquad g=k$$

Finally, evaluate $f(a,b)$ at each critical point (a,b) of F.

Example. The highway department is planning to build a picnic area for motorists along a major highway. It is to be rectangular with an area of 5000 m^2 and is to be fenced off on the three sides not adjacent to the highway. What is the least amount of fencing that will be needed to complete the job?



Solution. Label the sides of the picnic area as in the figure and let f denote the amount of fencing required.

Then, $f(x,y) = x + 2y$, and clearly, $x, y > 0$.
The goal is to minimize f given the requirement that the area must be 5000 m^2, that is, subject to the constraint
$g(x,y) = xy = 5000$.

Find the partial derivatives:
$f_x = 1$, $f_y = 2$, $g_x = y$, and $g_y = x$.

Obtain the three Lagrange equations:
$1 = \lambda y$, $2 = \lambda x$, and $xy = 5000$.

From the first and second equations,
$\lambda = \frac{1}{y}$ and $\lambda = \frac{2}{x}$ (since $y \neq 0$ and $x \neq 0$) which implies that
$\frac{1}{y} = \frac{2}{x}$ or $x = 2y$.

Now, substitute $x = 2y$ into the third Lagrange equation:
$2y^2 = 5000$ or $y = \pm 50$, but use the positive value for length!

Substitute $y = 50$ into the equation $x = 2y$ to get that $x = 100$.

Thus, the total amount of fencing needed is
$100 + 50 + 50 = 200$ m.

Example. Find the maximum and minimum values of the function $f(x,y) = xy$ subject to the constraint $x^2 + y^2 = 8$.

Solution. In this case, $g(x,y) = x^2 + y^2$.

Find the partial derivatives:
$f_x = y$, $f_y = x$, $g_x = 2x$, and $g_y = 2y$.

Obtain the three Lagrange equations:
$y = 2\lambda x$, $x = 2\lambda y$, and $x^2 + y^2 = 8$, note that neither x nor y can be zero.

Rewrite the first two equations as
$2\lambda = \frac{y}{x}$ and $2\lambda = \frac{x}{y}$ which implies that
$\frac{y}{x} = \frac{x}{y}$ or $y^2 = x^2$.

Now, substitute $y^2 = x^2$ into the third equation to get
$2x^2 = 8$ or $x = \pm 2$.

If $x = 2$, it follows from equation $x^2 = y^2$ that $y = 2$ or $y = -2$.
If $x = -2$, it follows from equation $x^2 = y^2$ that $y = 2$ or $y = -2$.

Hence, there are four points at which the constrained extrema can occur:
$(2,2)$, $(2,-2)$, $(-2,2)$, and $(-2,-2)$.

Since $f(2,2) = f(-2,-2) = 4$ and $f(2,-2) = f(-2,-2) = -4$, it follows that when $x^2 + y^2 = 8$,
the maximum value of $f(x,y)$ is 4 at the points $(2,2)$ and $(2,-2)$,
and the minimum value of $f(x,y)$ is -4 at the points $(-2,2)$ and $(-2,-2)$.