

Partial Fraction Decomposition for Integration

If $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is less than the degree of $Q(x)$, then it follows from a theorem in algebra that:

$$\frac{P(x)}{Q(x)} = F_1 + F_2 + \dots + F_n \quad \text{where each } F_i \text{ is of the form}$$
$$\frac{A}{(ax + b)^m} \text{ or } \frac{Bx + C}{(ax^2 + bx + c)^p} \text{ for}$$

some positive integers m and p and $a \neq 0$

Suppose that the integral has a rational function. First, using long division or otherwise, reduce the rational function to the case $\frac{P(x)}{Q(x)}$, where the degree of $P(x)$ is less than the degree of $Q(x)$, so that the above result can be used.

Next, factor $Q(x)$ as much as possible (i.e., until obtain linear or irreducible quadratic factors are obtained), and using the above, write a partial fraction decomposition. Solve for all constants involved, and integrate.

Partial Fraction Decomposition of Rational Expressions

Examples. Focus on decomposing the denominator.

$$\frac{-5}{x^2(x+2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}$$

$$\frac{3x+4}{(2x-1)^2(x^2+1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$\begin{aligned} \frac{-x^3+2x-10}{(x-2)(x^2-4)(x^2+9)^2} &= \frac{-x^3+2x-10}{(x-2)(x-2)(x+2)(x^2+9)^2} \\ &= \frac{-x^3+2x-10}{(x-2)^2(x+2)(x^2+9)^2} \\ &= \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{Dx+E}{x^2+9} + \frac{Fx+G}{(x^2+9)^2} \end{aligned}$$

Example. Evaluate $\int \frac{4x^3 - 3x^2 + 6x - 27}{x^2(x^2 + 9)} dx$.

Solution. Let $f(x) = \frac{4x^3 - 3x^2 + 6x - 27}{x^2(x^2 + 9)}$. Note that the degree in the numerator is less than the degree in the denominator.

Factor the denominator (done!), and then decompose the rational expression into partial fractions.

Identify the highest degree of x in the above equation: x^3

$$\frac{4x^3 - 3x^2 + 6x - 27}{x^2(x^2 + 9)} = \boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9}}$$

Once the constants A , B , and C are found, the boxed piece on the right-hand side of the equation will be integrated.

To solve for the constants:

Multiply the above equation by the common denominator $x^2(x^2 + 9)$, and simplify the expression.

$$4x^3 - 3x^2 + 6x - 27 = x^2(x^2 + 9) \left(\frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9} \right)$$

$$4x^3 - 3x^2 + 6x - 27 = Ax(x^2 + 9) + B(x^2 + 9) + (Cx + D)x^2$$

$$4x^3 - 3x^2 + 6x - 27 = A(x^3 + 9x) + B(x^2 + 9) + Cx^3 + Dx^2$$

Compare the coefficients of x on both sides of the equation:

$$x^3: \quad 4 = A + C$$

$$x^2: \quad -3 = B + D$$

$$x^1: \quad 6 = 9A$$

$$x^0: \quad -27 = 9B$$

Right away, $A = \frac{2}{3}$ and $B = -3$. Solve the system of equations to get

$$C = \frac{10}{3} \text{ and } D = 0.$$

Finally, substituting the values found for the constants in the boxed piece, the integral becomes

$$\int \frac{4x^3 - 3x^2 + 6x - 27}{x^2(x^2 + 9)} dx = \int \left(\frac{2}{3x} - \frac{3}{x^2} + \frac{10x}{3x^2 + 9} \right) dx$$

Using integration by substitution for the last integral,

$$\int \left(\frac{2}{3x} - \frac{3}{x^2} + \frac{10x}{3x^2 + 9} \right) dx = \frac{2}{3} \ln|x| + \frac{3}{x} + \frac{5}{3} \ln|x^2 + 9| + C$$