The Robert Gillespie ACADEMIC SKILLS CENTRE

Derivative of a Function

Derivative of a Function $f(x)$ at a Number a

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}, \text{ or } f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a},$$

provided that the limit exists. (Note that the former formula is used more often than the latter.)

Example. Using the definition of the derivative at a number, find the slope of tangent line at $x=-2$ to the curve $y=-6x^2$. What is the equation of the tangent line at this point?

Solution. By definition of the derivative, the slope of the tangent line at $x=-2$ is

$$\begin{aligned} f'(-2) &= \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h} \\ &= \lim_{h\to 0}\frac{-6(-2+h)^2-(-6(-2)^2)}{h} \\ &= \lim_{h\to 0}\frac{-24+24h-6h^2+24}{h} \\ &= \lim_{h\to 0}\frac{h(24-6h)}{h} = \lim_{h\to 0}(24-6h) = 24 \end{aligned}$$

When $x=-2$, $y(-2)=-6(-2)^2=-24$, that is, the point of tangency is $(-2,-24)$.

Using the point-slope formula,
$$\begin{aligned} y+24 &= 24(x+2) \\ y &= 24x+24 \end{aligned}.$$

Derivative Function $f'(x)$ of a Function $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$, for any x for which this limit exists.

Example. Using the definition of the derivative, given the function $f(x) = 3 - \frac{1}{x^2}$, find $f'(x)$.

Solution. By the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[3 - \frac{1}{(x+h)^2}\right] - \left[3 - \frac{1}{x^2}\right]}{h} = \lim_{h \to 0} \frac{-\frac{1}{(x+h)^2} + \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-x^2 + (x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h \to 0} \left(\frac{-x^2 + x^2 + 2xh + h^2}{x^2(x+h)^2}\right)\left(\frac{1}{h}\right)$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{hx^2(x+h)^2} = \lim_{h \to 0} \frac{2x+h}{x^2(x+h)^2} = \frac{2x}{x^2(x)^2} = \frac{2}{x^3}$$