

TORI IN SYMPLECTOMORPHISM GROUPS

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Finiteness theorem: *Let (M, ω) be a four dimensional compact symplectic manifold and $T \cong (S^1)^2$ a two dimensional torus. Then the set of effective Hamiltonian T -actions on (M, ω) modulo equivariant symplectomorphisms and modulo automorphisms of T is finite.*

Remarks.

- (1) This is a “95% theorem”, as its complete proof has not yet been \LaTeX ed.
- (2) The image of T in the symplectomorphism group $\text{Sympl}(M, \omega)$ is a maximal torus. This follows from the fact that the orbits of a Hamiltonian torus action are isotropic.
- (3) If (M, ω) admits a Hamiltonian T -action then every symplectic T -action on (M, ω) is Hamiltonian. In this case, the theorem asserts that the number of conjugacy classes of two-dimensional tori in $\text{Sympl}(M, \omega)$ is finite.
- (4) In contrast, Eugene Lerman has constructed a compact contact manifold that admits infinitely many non-conjugate toric actions.

As a consequence of a theorem of Delzant, a Hamiltonian T -action on M with moment map Φ is equivariantly symplectomorphic to the symplectic toric manifold $(M_\Delta, \omega_\Delta)$ associated to the Delzant polygon $\Delta = \Phi(M) \subset \mathfrak{t}^*$. We need to show that the number of Delzant polygons Δ such that $(M_\Delta, \omega_\Delta)$ is symplectomorphic to (M, ω) , modulo translations and $\text{GL}(2, \mathbb{Z})$ -congruence, is finite.

The “size” of an edge of a Delzant polygon is measured by its “rational length”, which is characterized by being invariant under $\text{GL}(2, \mathbb{Z})$ -congruence and translations and being standard along the coordinate axes. The moment map preimage of an edge is a symplectic sphere whose symplectic area is 2π times the rational length of the edge.

Examples of Delzant polygons are a “Delzant triangle”, which corresponds to $\mathbb{C}\mathbb{P}^2$, and a “Hirzebruch trapezoid”, which corresponds to a Hirzebruch surface. See Figure 1. Up to translations and $\text{GL}(2, \mathbb{Z})$ -congruence, a Delzant triangle is determined by the rational length λ of each side, and a Hirzebruch trapezoid is determined by parameters (a, b, k) where b is its height, a is the average of the lengths of its top and bottom edges, and k is a non-negative integer such that the right edge has slope $-1/k$ (or is vertical if $k = 0$). A Hirzebruch surface is a $\mathbb{C}\mathbb{P}^1$ bundle over $\mathbb{C}\mathbb{P}^1$. The moment map preimages of the top and bottom edges are the

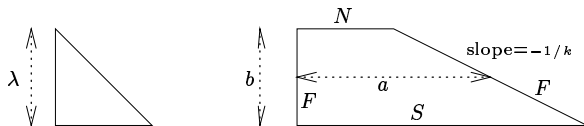


FIGURE 1. Delzant triangle and Hirzebruch trapezoid

“north pole section” and the “south pole section”; the moment map preimages of the side edges are fibers.

The perimeter and area of a Delzant polygon Δ are symplectic invariants of the underlying toric variety $(M_\Delta, \omega_\Delta)$: the perimeter is equal to the pairing of ω_Δ with the first Chern class $c_1(TM_\Delta)$, and the area is equal to the Liouville volume $\frac{1}{2\pi} \int_M \omega_\Delta^2 / 2!$.

An equivariant symplectic blowup of size δ of a toric manifold amounts to “chopping” off a corner of size δ of its polygon. This reduces the perimeter by δ and the area by $\frac{1}{2}\delta^2$. The preimage of the new edge is the exceptional divisor. A homology class which is represented by the moment map preimage of an edge gives a homology class in the blown up manifold which is represented by the preimage of at most two edges of the “chopped” polygon. After s blowups, the symplectic area of such a homology class is bounded by 2^s times the perimeter.

Each Delzant polygon is either a Delzant triangle or is obtained from a Hirzebruch trapezoid by a sequence of “corner choppings”, so each symplectic toric manifold is either $\mathbb{C}\mathbb{P}^2$ or is obtained from a Hirzebruch surface by a sequence of equivariant symplectic blow-ups.

Fix a symplectic manifold (M, ω) . To prove the finiteness theorem for this manifold, it is enough to show that the number of tuples $(a, b, k; \delta_1, \dots, \delta_s)$ such that (M, ω) is symplectomorphic to a symplectic toric manifold $(M_\Delta, \omega_\Delta)$ that is obtained from a Hirzebruch surface with parameters (a, b, k) by equivariant symplectic blow-ups of sizes $\delta_1, \dots, \delta_s$ is finite.

Suppose that (M, ω) is symplectomorphic to a toric manifold $(M_\Delta, \omega_\Delta)$ that is obtained from a Hirzebruch surface with parameters (a, b, k) by a sequence of equivariant symplectic blowups of sizes $\delta_1, \dots, \delta_s$. Let

$$E_1, \dots, E_s \in H_2(M)$$

be the homology classes of the exceptional divisors. Then

- (1) $E_i \cdot E_i = -1$;
- (2) E_i can be represented by an embedded symplectic sphere;
- (3) $\langle \omega, E_i \rangle$ is smaller than 2^s times $\langle \omega, c_1(TM) \rangle$.

As a consequence of Gromov’s compactness, there exist only finitely many cohomology classes with these properties. Because $\delta_i = \langle \omega, E_i \rangle$, the set of possible s -tuples $(\delta_1, \dots, \delta_s)$ is finite.

The perimeter of the Delzant polygon Δ is $2(a + b) - \sum_{j=1}^s \delta_j$ and its area is $ab - \frac{1}{2} \sum_{j=1}^s \delta_j^2$. Because these are symplectic invariants of (M, ω) , we can recover $a + b$ and ab from $\delta_1, \dots, \delta_s$, so the set of possible values for a and b is finite. Let

$$N, S, F \in H_2(M)$$

be the homology classes coming from the north pole section, south pole section, and fiber, respectively. Then

- (1) $S = N + kF$;
- (2) $\langle \omega, S \rangle$ is smaller than 2^s times $\langle \omega, c_1(TM) \rangle$;
- (3) $\langle \omega, N \rangle$ is positive;
- (4) $\langle \omega, F \rangle = b$.

It follows that the non-negative integer k is bounded from above by $2^s \langle \omega, c_1(TM) \rangle / b$. Because there are finitely many possibilities for the value of b , there are finitely

many possibilities for the value of k . This completes the outline of the proof of the finiteness theorem.

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