

## TORI IN SYMPLECTOMORPHISM GROUPS

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**Finiteness theorem:** *Let  $(M, \omega)$  be a four dimensional compact symplectic manifold and  $T \cong (S^1)^2$  a two dimensional torus. Then the set of effective Hamiltonian  $T$ -actions on  $(M, \omega)$  modulo equivariant symplectomorphisms and modulo automorphisms of  $T$  is finite.*

*Remarks.*

- (1) This is a “95% theorem”, as its complete proof has not yet been  $\LaTeX$ ed.
- (2) The image of  $T$  in the symplectomorphism group  $\text{Sympl}(M, \omega)$  is a maximal torus. This follows from the fact that the orbits of a Hamiltonian torus action are isotropic.
- (3) If  $(M, \omega)$  admits a Hamiltonian  $T$ -action then every symplectic  $T$ -action on  $(M, \omega)$  is Hamiltonian. In this case, the theorem asserts that the number of conjugacy classes of two-dimensional tori in  $\text{Sympl}(M, \omega)$  is finite.
- (4) In contrast, Eugene Lerman has constructed a compact contact manifold that admits infinitely many non-conjugate toric actions.

As a consequence of a theorem of Delzant, a Hamiltonian  $T$ -action on  $M$  with moment map  $\Phi$  is equivariantly symplectomorphic to the symplectic toric manifold  $(M_\Delta, \omega_\Delta)$  associated to the Delzant polygon  $\Delta = \Phi(M) \subset \mathfrak{t}^*$ . We need to show that the number of Delzant polygons  $\Delta$  such that  $(M_\Delta, \omega_\Delta)$  is symplectomorphic to  $(M, \omega)$ , modulo translations and  $\text{GL}(2, \mathbb{Z})$ -congruence, is finite.

The “size” of an edge of a Delzant polygon is measured by its “rational length”, which is characterized by being invariant under  $\text{GL}(2, \mathbb{Z})$ -congruence and translations and being standard along the coordinate axes. The moment map preimage of an edge is a symplectic sphere whose symplectic area is  $2\pi$  times the rational length of the edge.

Examples of Delzant polygons are a “Delzant triangle”, which corresponds to  $\mathbb{C}\mathbb{P}^2$ , and a “Hirzebruch trapezoid”, which corresponds to a Hirzebruch surface. See Figure 1. Up to translations and  $\text{GL}(2, \mathbb{Z})$ -congruence, a Delzant triangle is determined by the rational length  $\lambda$  of each side, and a Hirzebruch trapezoid is determined by parameters  $(a, b, k)$  where  $b$  is its height,  $a$  is the average of the lengths of its top and bottom edges, and  $k$  is a non-negative integer such that the right edge has slope  $-1/k$  (or is vertical if  $k = 0$ ). A Hirzebruch surface is a  $\mathbb{C}\mathbb{P}^1$  bundle over  $\mathbb{C}\mathbb{P}^1$ . The moment map preimages of the top and bottom edges are the

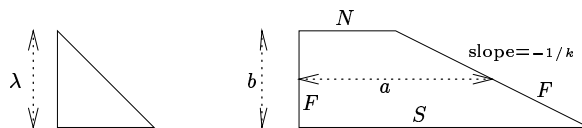


FIGURE 1. Delzant triangle and Hirzebruch trapezoid

“north pole section” and the “south pole section”; the moment map preimages of the side edges are fibers.

The perimeter and area of a Delzant polygon  $\Delta$  are symplectic invariants of the underlying toric variety  $(M_\Delta, \omega_\Delta)$ : the perimeter is equal to the pairing of  $\omega_\Delta$  with the first Chern class  $c_1(TM_\Delta)$ , and the area is equal to the Liouville volume  $\frac{1}{2\pi} \int_M \omega_\Delta^2 / 2!$ .

An equivariant symplectic blowup of size  $\delta$  of a toric manifold amounts to “chopping” off a corner of size  $\delta$  of its polygon. This reduces the perimeter by  $\delta$  and the area by  $\frac{1}{2}\delta^2$ . The preimage of the new edge is the exceptional divisor. A homology class which is represented by the moment map preimage of an edge gives a homology class in the blown up manifold which is represented by the preimage of at most two edges of the “chopped” polygon. After  $s$  blowups, the symplectic area of such a homology class is bounded by  $2^s$  times the perimeter.

Each Delzant polygon is either a Delzant triangle or is obtained from a Hirzebruch trapezoid by a sequence of “corner choppings”, so each symplectic toric manifold is either  $\mathbb{C}\mathbb{P}^2$  or is obtained from a Hirzebruch surface by a sequence of equivariant symplectic blow-ups.

Fix a symplectic manifold  $(M, \omega)$ . To prove the finiteness theorem for this manifold, it is enough to show that the number of tuples  $(a, b, k; \delta_1, \dots, \delta_s)$  such that  $(M, \omega)$  is symplectomorphic to a symplectic toric manifold  $(M_\Delta, \omega_\Delta)$  that is obtained from a Hirzebruch surface with parameters  $(a, b, k)$  by equivariant symplectic blow-ups of sizes  $\delta_1, \dots, \delta_s$  is finite.

Suppose that  $(M, \omega)$  is symplectomorphic to a toric manifold  $(M_\Delta, \omega_\Delta)$  that is obtained from a Hirzebruch surface with parameters  $(a, b, k)$  by a sequence of equivariant symplectic blowups of sizes  $\delta_1, \dots, \delta_s$ . Let

$$E_1, \dots, E_s \in H_2(M)$$

be the homology classes of the exceptional divisors. Then

- (1)  $E_i \cdot E_i = -1$ ;
- (2)  $E_i$  can be represented by an embedded symplectic sphere;
- (3)  $\langle \omega, E_i \rangle$  is smaller than  $2^s$  times  $\langle \omega, c_1(TM) \rangle$ .

As a consequence of Gromov’s compactness, there exist only finitely many cohomology classes with these properties. Because  $\delta_i = \langle \omega, E_i \rangle$ , the set of possible  $s$ -tuples  $(\delta_1, \dots, \delta_s)$  is finite.

The perimeter of the Delzant polygon  $\Delta$  is  $2(a + b) - \sum_{j=1}^s \delta_j$  and its area is  $ab - \frac{1}{2} \sum_{j=1}^s \delta_j^2$ . Because these are symplectic invariants of  $(M, \omega)$ , we can recover  $a + b$  and  $ab$  from  $\delta_1, \dots, \delta_s$ , so the set of possible values for  $a$  and  $b$  is finite. Let

$$N, S, F \in H_2(M)$$

be the homology classes coming from the north pole section, south pole section, and fiber, respectively. Then

- (1)  $S = N + kF$ ;
- (2)  $\langle \omega, S \rangle$  is smaller than  $2^s$  times  $\langle \omega, c_1(TM) \rangle$ ;
- (3)  $\langle \omega, N \rangle$  is positive;
- (4)  $\langle \omega, F \rangle = b$ .

It follows that the non-negative integer  $k$  is bounded from above by  $2^s \langle \omega, c_1(TM) \rangle / b$ . Because there are finitely many possibilities for the value of  $b$ , there are finitely

many possibilities for the value of  $k$ . This completes the outline of the proof of the finiteness theorem.

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