LOCALIZATION FOR HAMILTONIAN GROUP ACTIONS

MEGUMI HARADA AND YAEL KARSHON

Let Δ be a compact convex polytope, i.e., the convex hull of a finite set of points in $\mathbb{R}^n$. The *tangent cone* at a face $F \subset \Delta$ is $C_F = \{x + r(y - x) \mid x \in F \ , \ y \in \Delta \ , \ r \geq 0\}$. The polytope is *simple* if exactly n facets meet at each vertex. The Lawrence-Varchenko formula expresses the characteristic function of a simple polytope as an alternating sum of the characteristic functions of convex polyhedral cones. Each of these cones is obtained from the tangent cone at a vertex by flipping some of its extremal rays and removing the corresponding facets. We think of the Lawrence-Varchenko formula for polytopes as a combinatorial counterpart of localization formulas for Hamiltonian torus actions: When applied to the moment polytope of a toric manifold, the Lawrence-Varchenko formula gives the Guillemin-Lerman-Sternberg ("G-L-S") formula for the toric manifold. The G-L-S formula is obtained by Fourier transform from the Duistermaat-Heckman exact stationary phase formula, which, in turn, is a special case of the Berline-Vergne localization in equivariant cohomology.

Another formula, due to Brianchon and Gram, expresses the characteristic function of an arbitrary compact convex polytope as an alternating sum of the characteristic functions of the tangent cones of its faces (in all dimensions):

$$\mathbf{1}_\Delta(\cdot) = \sum_F (-1)^{\dim F} \mathbf{1}_{C_F}(\cdot).$$

Shlomo Sternberg asked: What is a localization formula for manifolds that, when applied to a toric manifold, will yield the Brianchon-Gram decomposition of its moment polytope?

Paradan-Woodward, following Witten, gave a formula for the integral of an equivariant cohomology class as a summation over critical sets of the norm squared of the moment map. For a toric manifold, these critical sets exactly correspond to feet of the perpendiculars from the origin to the different faces of the polytope. A relation to the Brianchon-Gram formula (that is valid for many but not all toric manifolds) was observed independently by several people, including us and Jonathan Weitsman. Jose Agapito and Leonor Godinho worked out this correspondence and, as a result, found new polytope decompositions.

Harada and Karshon developed a new localization formula for the Duistermaat-Heckman measure that can yield the Brianchon-Gram decomposition for *all* toric manifolds. (Warning: the details are not yet completely LaTeXed.) As in earlier work of Ginzburg-Guillemin-Karshon ("G-G-K"), this is done through cobordisms of non-compact spaces. The additional data which gives us control on these cobordisms in spite of their non-compactness is more general than in G-G-K: inspired by Maxim Braverman, we fix a G-equivariant bounded function

$$v \colon M \to \mathfrak{g}$$

and insist that the v-component Φ^v of our moment maps be proper and bounded from below.

We hope that this new approach will also lead to a common generalization of the Berline-Vergne and the Paradan-Woodward formulas.