

APPENDIX

In this appendix we will describe some explicit symplectic embeddings of balls into $\mathbb{C}P^n$. Recall, the Fubini-Study form σ_n on $\mathbb{C}P^n$ is normalized so that $\int_{\mathbb{C}P^1} \sigma_n = \pi$, and we denote by $B^{2n}(r)$ the closed ball of radius r in \mathbb{C}^n .

By Gromov [G1], if we can embed the disjoint union $B^{2n}(r_1) \sqcup B^{2n}(r_2)$ in $\mathbb{C}P^n$ symplectically then the radii must satisfy $r_1^2 + r_2^2 < 1$. Here we will embed $n + 1$ balls in $\mathbb{C}P^n$ whenever their radii r_0, \dots, r_n satisfy

$$r_i^2 + r_j^2 < 1 \quad \text{for all } i \neq j, \quad (1)$$

thus Gromov's condition is both necessary and sufficient in this case.

Gromov's condition (1) implies that for $k \geq 2$, if we can embed k symplectic balls of radius r in $\mathbb{C}P^n$ then $2r^2 \leq 1$. Because $\text{vol } \mathbb{C}P^n = \text{vol } B^{2n}(1)$ in our normalization, and because $\text{vol}(\sqcup_k B^{2n}(r)) = kr^{2n} \text{vol } B^{2n}(1)$; the portion of $\mathbb{C}P^n$ which can be symplectically filled by k equal balls is at most $k/2^n$. In the notation of §1.1;

$$v(\mathbb{C}P^n, k) \leq \frac{k}{2^n}. \quad (2)$$

Our construction implies that the bound (2) is tight for $1 \leq k \leq n + 1$.

The construction. We think of $\mathbb{C}P^n$ as the quotient of the unit sphere in \mathbb{C}^{n+1} by the diagonal action of S^1 , so when we write homogeneous coordinates $[u_0 : \dots : u_n]$ we will always assume that

$$|u_0|^2 + \dots + |u_n|^2 = 1. \quad (3)$$

In fact, let $i : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ be the standard inclusion and $\pi : S^{2n+1} \rightarrow \mathbb{C}P^n$ the standard projection. Let ω_n be the standard symplectic form on \mathbb{C}^n , then

$$\pi^* \sigma_n = i^* \omega_{n+1}. \quad (4)$$

This can be taken as the definition of σ_n .

We define a map $f_0 : B^{2n}(1) \rightarrow \mathbb{C}P^n$ by

$$f_0 : (z_1, \dots, z_n) \mapsto [\sqrt{1 - |\mathbf{z}|^2} : z_1 : \dots : z_n]. \quad (5)$$

where $|\mathbf{z}|^2 = |z_1|^2 + \dots + |z_n|^2$. We now show, following [McD3, 2.7.1], that this map is a symplectomorphism of the open unit ball in \mathbb{C}^n with $\mathbb{C}P^n \setminus S$, where $S = \{[0 : u_1 : \dots : u_n]\} \cong \mathbb{C}P^{n-1}$. First note that $f_0 = \pi \circ \tilde{f}_0$ where $\tilde{f}_0 : \mathbb{C}^n \rightarrow S^{n+1}$ is defined by the same formula as f_0 . One can easily check that $\tilde{f}_0^* \omega_{n+1} = \omega_n$. Thus $f_0^* \sigma_n = \tilde{f}_0^* \pi^* \sigma_n = \tilde{f}_0^* i^* \omega_{n+1} = \omega_n$.

Let $r < 1$, then the image under f_0 of the smaller ball $B^{2n}(r)$ is $\{[u_0 : \dots : u_n] \mid |u_1|^2 + \dots + |u_n|^2 \leq r^2\} = \{[u_0 : \dots : u_n] \mid |u_0|^2 \geq 1 - r^2\}$.

Similarly, we have embeddings f_1, \dots, f_n where

$$f_1 : (z_0, z_2, \dots, z_n) \mapsto [z_0 : \sqrt{1 - |z|^2} : z_2 : \dots : z_n]$$

and so on. We embed $B^{2n}(r_j)$ into $\mathbb{C}P^n$ by f_j ; its image is

$$B_j = \{[u_0 : \dots : u_n] \mid |u_j|^2 \geq 1 - r_j^2\}.$$

We claim that the B_j are disjoint if the r_j 's satisfy (1). Indeed, suppose that $[u_0 : \dots : u_n]$ belongs to both B_i and B_j , then $|u_i|^2 \geq 1 - r_i^2$ and $|u_j|^2 \geq 1 - r_j^2$, but then $|u_i|^2 + |u_j|^2 \geq 2 - (r_i^2 + r_j^2) > 1$ which contradicts (3).

Connection with the moment map. The above embeddings were originally found with the help of the *moment map*. We will illustrate this for the case of $\mathbb{C}P^2$. The same method gives embeddings of balls in any $\mathbb{C}P^n$ and more generally in any *symplectic toric variety*; for example in some blow-ups of $\mathbb{C}P^n$'s.

Suppose that a torus T acts on a symplectic manifold (M, ω) . Let \mathfrak{t}^* be the dual of the Lie algebra of T . For any $\xi \in \mathfrak{t}$ let ξ_M be the vector field on M which generates the flow $p \mapsto \exp(t\xi)(p)$. A *moment map* for (M, ω, T) is a map $\Phi : M \rightarrow \mathfrak{t}^*$ such that for all $\xi \in \mathfrak{t}$,

$$\langle d\Phi, \xi \rangle = i(\xi_M)\omega. \quad (6)$$

By [A, G-S 2], if M is compact then $\text{image}(\Phi)$ is a convex polytope in \mathfrak{t}^* . Suppose now that $\dim T = 1/2 \dim M$ and that T acts effectively. Then by a theorem of Delzant [Del], the polytope $\text{image}(\Phi)$ determines (M, ω, T) up to an equivariant symplectomorphism.

Delzant's theorem also works locally. In fact, suppose that (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds of dimension $2n$, with effective actions of a torus $T = T^n$ and with moment maps $\Phi_j : M_j \rightarrow \mathfrak{t}^*$ which are proper. Let $U \subseteq \mathfrak{t}^*$ be a convex set such that $\text{image}(\Phi_1) \cap U = \text{image}(\Phi_2) \cap U$. Then there is an equivariant symplectomorphism $F : \Phi_1^{-1}U \rightarrow \Phi_2^{-1}U$ with $\Phi_1 = \Phi_2 \circ F$. To see this, first refer to [C-D-M] for an explanation as to how the properness of Φ_j can replace the compactness of M for all practical purposes, and then apply Delzant's proof.

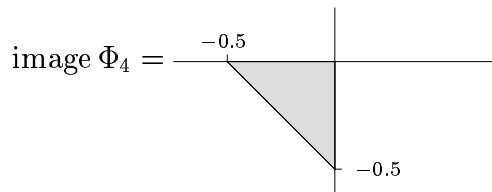
Delzant's theorem is extremely useful in that it lets us visualize a manifold of dimension $2n$ as a polytope in \mathbb{R}^n . Every point p in the polytope represent a T orbit in the manifold, namely, the preimage of p . Information about the manifold can often be read from the polytope.

Example 7. Take the standard symplectic form on \mathbb{C}^2 ; $\omega_2 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. The torus $T^2 = S^1 \times S^1$ acts by $(\lambda_1, \lambda_2) \cdot (z_1, z_2) = (\lambda_1 z_1, \lambda_2 z_2)$ where $z_j = x_j + iy_j$, with a moment map $\Phi_1(z_1, z_2) = (-|z_1|^2/2, -|z_2|^2/2)$. Its image is the negative quadrant in \mathbb{R}^2 . The image of the ball $B^4(r)$ is the triangle $\{(x, y) \mid x \leq 0, y \leq 0, x + y \geq -r^2/2\}$.

Example 8. T^2 acts on (\mathbb{C}^2, ω_2) by $(\lambda_1, \lambda_2) \cdot (z_1, z_2) = (\lambda_1^{-1} z_1, \lambda_1^{-1} \lambda_2 z_2)$, with a moment map $\Phi_2(z_1, z_2) = (-1/2 + |z_1|^2/2 + |z_2|^2/2, -|z_2|^2/2)$.

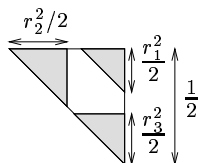
Example 9. T^2 acts on (\mathbb{C}^2, ω_2) by $(\lambda_1, \lambda_2) \cdot (z_1, z_2) = (\lambda_2^{-1} z_1, \lambda_1 \lambda_2^{-1} z_2)$, with a moment map $\Phi_3(z_1, z_2) = (-|z_2|^2/2, -1/2 + |z_1|^2/2 + |z_2|^2/2)$.

Example 10. T^2 acts on $\mathbb{C}P^2$ by $(\lambda_1, \lambda_2) \cdot [z_0 : z_1 : z_2] = [z_0 : \lambda_1 z_1 : \lambda_2 z_2]$ with a moment map $\Phi_4[z_0 : z_1 : z_2] = (-1/2|z_1|^2, -1/2|z_2|^2)$ if $|z_0|^2 + |z_1|^2 + |z_2|^2 = 1$.



This can be seen in the following way. By (4), it is sufficient to show that (6) holds for the function $\Phi_5 : \mathbb{C}^3 \rightarrow \mathbb{R}^2$ which is given by $\Phi_5(z_0, z_1, z_2) = (-1/2|z_1|^2, -1/2|z_2|^2)$ and for the T^2 action $(\lambda_1, \lambda_2) \cdot (z_0, z_1, z_2) = (z_0, \lambda_1 z_1, \lambda_2 z_2)$ and for the standard 2-form ω_3 . This itself is a very easy computation.

The images of the moment maps in examples 7 and 10 look alike in a neighborhood of the origin in \mathbb{R}^2 . Similarly, the images in examples 8 and 10 look alike near the vertex $(-1/2, 0)$, and in examples 9 and 10 – near the vertex $(0, -1/2)$. We now apply Delzant's theorem three times, to identify neighborhoods of $[1 : 0 : 0]$, of $[0 : 1 : 0]$ and of $[0 : 0 : 1]$ in $\mathbb{C}P^2$ with a neighborhood of 0 in \mathbb{C}^2 . The following picture shows three balls of radii r_1, r_2, r_3 embedded in $\mathbb{C}P^2$ via these identifications, as viewed in the image of the moment map.



Connection to other embeddings. In [McD3], McDuff describes an explicit symplectic embedding of two disjoint balls into $\mathbb{C}P^2$. She

does this by using metric balls in $\mathbb{C}P^2$ with respect to the Fubini-Study metric. She notes that a metric ball of radius μ is also a symplectic ball of radius r where $r = \sin \mu$; the condition (5) then becomes $\mu_1 + \mu_2 < \pi/2$ and this condition allows her to find two disjoint metric balls in $\mathbb{C}P^2$ with radii μ_1, μ_2 . One of them is centered at $[1 : 0 : 0]$ and the other is centered at the plane $S = \{[0 : z : w]\}$. Our embedding (5) coincides with McDuff's [McD3, 2.7.1]. In particular, our balls are also metric balls. One of our balls is centered at $[1 : 0 : 0]$ and the other two are centered at S .

REFERENCES

- [A] M. Atiyah, *Convexity and commuting hamiltonians*, Bull. London Math. Soc. **14** (1982), 1-15.
- [C-D-M] M. Condevaux, P. Dazord and P. Molino, *Géométrie du Moment (Séminaire Sud-Rhodanien)*, Publications du département de mathématiques, Université Claude Bernard-Lyon I, 1988.
- [Del] T. Delzant, *Hamiltoniens périodiques et image convexe de l'application moment*, Bulletin de la Société Mathématique de France **116** (1988), 315-339.
- [McD3] D. McDuff, *Remarks on the uniqueness of symplectic blowing up*,
- [G1] M. Gromov, *Pseudo-holomorphic curves in almost complex manifolds*, Invent. Math. **82** (1985), 307-347.
- [G-S 2] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491-513.