HAMILTONIAN TORUS ACTIONS

YAEL KARSHON

These are notes from my lecture at the conference *Geometry and Physics* at Aarhus University, July 1995. Part 1 is an introduction to Hamiltonian T-spaces. Part 2 is on cobordisms and cohomological invariants; this material will appear in a joint publication with Victor Guillemin and Viktor Ginzburg [GGK]. Part 3, on "x-rays", outlines joint work with Susan Tolman which we are currently writing up [KT]. In the lecture itself I skipped most of part 3 for lack of time.

This lecture is about various equivalence relations between Hamiltonian T-spaces: isomorphism, cobordism, local isomorphism.

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PART 1 : INTRODUCTION

Definitions. Let $T = S^1 \times \ldots \times S^1$ be a k-dimensional torus and $\mathfrak{t}$ its Lie algebra. The dual vector space, $\mathfrak{t}^*$, is isomorphic to $\mathbb{R}^k$ and contains the integral weight lattice, $\mathbb{Z}^k \subset \mathbb{R}^k$. Let M be a smooth manifold and let $T \times M \longrightarrow M$ be a smooth action of the torus on this manifold. We assume that the action is effective, i.e., that the group homomorphism $T \longrightarrow \text{Diff}(M)$ is one to one. Let ω be a symplectic form on M, i.e., a differential 2-form which is closed and non-degenerate. We assume that the torus action is symplectic: $a^*\omega = \omega$ for all $a \in T$.

To make this symplectic torus action into a *Hamiltonian* torus action we need a *moment map*, i.e., a map

$$\Phi : M \longrightarrow \mathfrak{t}^*$$

which satisfies

$$d < \Phi, \xi > = -\iota(\xi_M)\omega \tag{1}$$

for every $\xi \in \mathfrak{t}$, where ξ_M is the vector field which generates the action of the one parameter group $\exp(s\xi)$, $s \in \mathbb{R}$.

The moment map serves not as an extra piece of data (it is determined, up to an additive constant, by the symplectic torus action) but as a convenient way of organizing the action data. The obstruction for the existence of a moment map lies in $H^1(M)$ because the right hand side of (1) is a closed 1-form.

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The triple (M, ω, Φ) determines the T-action and is called a *Hamiltonian T-space.*

One can define Hamiltonian actions of other Lie groups. We restrict to torus actions because they are easier and because many questions on actions of compact Lie groups reduce to questions on the actions of their maximal tori.

Motivation. I like to list the following three excuses for studying Hamiltonian group actions.

1. **Physics:** The notion of a symplectic manifold originates in classical mechanics. The phase space of a mechanical system with holonomic constraints is a symplectic manifold. Symmetries of the system give rise to conserved quantities, by Noether's theorem; for example, a rotational symmetry gives rise to conservation of angular momentum. In mathematical language, symmetries are actions of Lie groups and the conserved quantities are the components of a moment map. This is where the name "moment map" comes from.
2. **Representation theory:** Borel-Weil-Bott-Kirillov-Kostant give a geometric construction for the linear representations of compact Lie groups. The central players in this game are flag manifolds with Hamiltonian group actions.
3. **Topology; geometry:** Even if a space is not initially given a symplectic structure or a group action, it is often worthwhile to introduce such a structure in order to compute certain invariants of the space such as volume, Betti numbers, or the ring structure on cohomology. See for example [2], [3], [1], [4].

Remarks on Hamiltonian torus actions.

1. When studying Hamiltonian spaces we will sometimes make one or both of the following assumptions.
   (a) The manifold M is compact, or the moment map Φ is proper.
   (b) The set of fixed points is isolated
   The assumption of isolated fixed points on a compact manifold rules out the possibility of taking the product with an arbitrary symplectic manifold on which the group acts trivially.
2. One uses moment maps to define quotients in the symplectic category. (The naive quotient M/T doesn't work; it could be odd dimensional.) The level sets of the moment map are preserved under the torus action. One defines the reduced spaces ("symplectic quotients") by $M//T := \Phi^{-1}(a)/T$. The dimension of this quotient is typically $\dim M - 2\dim T$. In fact, in a Hamiltonian space the torus can be at most half the dimension of the manifold.
3. An important recent development in the field of Hamiltonian group actions is Tolman's negative answer to the **Kähler question**. This question states:

> *Let G be a compact Lie group. Does every compact Hamiltonian G-space with isolated fixed points admit a compatible Kähler structure?*
>
> (A compatible Kähler structure is a complex structure J which is G-invariant and such that $\omega(u, Jv)$ defines a positive definite metric.) See the papers by Tolman [T] and by Woodward [W].

Definition. The *deficiency* of a Hamiltonian T-space (M, ω, Φ) is the non-negative integer $\frac{1}{2}\dim M - \dim T$.

Delzant spaces. The most useful source of examples of Hamiltonian torus actions is Delzant's classification of the compact Hamiltonian T-spaces for which the dimension of the torus T is half the dimension of the manifold (i.e., the deficiency is zero). We call these *Delzant spaces.* Every Delzant space admits a complex structure which makes it into a Kähler toric variety. The image of the moment map, which is a convex polytope by the convexity theorem of Atiyah-Guillemin-Sternberg, determines the space up to isomorphism. This *moment polytope* can be identified with the quotient M/T and it provides a powerful means of visualizing the manifold.

For example, $M = \mathbb{CP}^2$ sits over a triangle: We have the Fubini Study symplectic form on $\mathbb{CP}^2$, the torus action $(a, b) \cdot [z_0, z_1, z_2] = [z_0, az_1, bz_2]$, and the moment map $\Phi = (\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2})$. The image of the moment map is the triangle $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$.

One can produce spaces of higher deficiency by taking a Delzant space and restricting the action to a sub-torus $H \subseteq T$. The H-moment map is the composition $\pi \circ \Phi$ where $\pi : \mathfrak{t}^* \longrightarrow \mathfrak{h}^*$ is the natural projection.

PART 2 : COBORDISMS AND COHOMOLOGICAL INVARIANTS

Cobordisms. Recall that a cobordism between two compact oriented manifolds M_1 and M_2 is a third compact oriented manifold W with boundary $M_1 \sqcup (-M_2)$, where $-M_2$ is the manifold M_2 with its orientation reversed. In the presence of an extra structure on the M_i's we would like the cobording manifold W to have this structure too and the inclusion maps $M_i \hookrightarrow W$ to respect the structure. For example, in the presence of a group action on the M_i's we require the same group to act on W and we require the maps $M_i \hookrightarrow W$ to be equivariant.

There is no hope of making W symplectic if the M_i's are – the dimension of W is odd. Instead, we require W to carry a closed 2-form which may be degenerate.

Every symplectic manifold has a compatible almost complex structure, $J : TM \longrightarrow TM$. Compatible means that $\omega(u, Jv)$ defines a positive definite metric. An odd dimensional manifold cannot carry an almost complex structure; instead, we require our cobording manifold W to carry a *stable* complex structure. We recall the definition: a stable complex structure on W is given by a complex structure on a vector bundle $TW \oplus \mathbb{R}^k$ for some k.

Two such J_i's are equivalent if there exists an isomorphism of complex vector bundles $(TW \oplus \mathbb{R}^{k_1}) \oplus \mathbb{C}^{r_1} \cong (TW \oplus \mathbb{R}^{k_2}) \oplus \mathbb{C}^{r_2}$ for some r_1 and r_2. Note that an almost complex structure determines a stable complex structure.

Definition. A cobordism between two compact Hamiltonian T-spaces (M_i, ω_i, Φ_i) is a compact oriented manifold W with boundary $M_1 \sqcup (-M_2)$ and with the following structure:

1. a T-action
2. an invariant closed 2-form
3. a moment map
4. an equivariant stable complex structure

such that the inclusion maps $M_i \hookrightarrow W$ respect these structures. To make sense of (3), note that in the definition of a moment map the closed invariant 2-form ω need not be symplectic. Later we will allow the M_i's and W to be orbifolds; the structures (1)–(4) make sense on orbifolds.

Cohomological invariants of a symplectic manifold. What can we do with a closed 2-form and a stable complex structure? We can define characteristic numbers; these are the numbers $\int_M P(\omega, \{c_i\})$ where P is a polynomial and where $c_i \in H^{2i}(M)$ are the Chern classes of the complex vector bundle $TM \oplus \mathbb{R}^k$. For example, for a symplectic manifold (M, ω) the characteristic number $\int_M \omega^n / n!$ is equal to the volume of M if $\dim M = n$ and is zero otherwise. *Cohomological invariants* of (M, ω, J) are invariants which only depend on the characteristic numbers, possibly on an infinite number of them. For example, the volume of a symplectic manifold is the cohomological invariant $\int_M \exp(\omega)$.

If ω is non-degenerate and we require J to be compatible with ω then the c_i's are *symplectic* invariants, so the cohomological invariants are invariants of the pair (M, ω). In the non-symplectic case there is no such compatibility assumption and we produce invariants of the triple (M, ω, J).

An interesting cohomological invariant is the *quantization* of a symplectic manifold in the sense that we now explain. If (M, ω, J) is Kähler and $L \longrightarrow M$ is a Hermitian holomorphic line bundle with curvature $-2\pi i \omega$ then one can define the quantization of (M, ω) to be the virtual vector space $Q = \sum_i (-1)^i H^i(M, \mathcal{O}_L)$, where $\mathcal{O}_L$ is the sheaf of holomorphic sections of L. In many cases this alternating sum is equal to the vector space of global holomorphic sections, $H^0(M, \mathcal{O}_L)$. The virtual dimension of Q is the Riemann Roch Hirzebruch number, $\int_M \exp(\omega) \operatorname{Todd}(\{c_i\})$. Here, if s_i are the elementary symmetric polynomials in the variables x_j then $\operatorname{Todd}(\{s_i\}) = \prod_j \frac{x_j}{1-e^{-x_j}}$. Note that if $x_j = 0$ then $\frac{x_j}{1-e^{-x_j}} = 1$.

On a non-Kähler manifold we can use an almost complex structure to define quantization. A line bundle L and an almost complex structure J, plus additional structures (connection, metric) which do not effect the final answer, give rise to an elliptic differential operator $\mathcal{D}$ (the "rolled-up twisted Dolbeault operator") whose index is given by the same formula as before:

$\dim\ker(\mathcal{D}) - \dim\operatorname{coker}(\mathcal{D}) = \int_M \exp(\omega)\,\mathrm{Todd}(\{c_i\})$. We define the quantization of (M,ω) to be the virtual vector space $[\ker(\mathcal{D})] - [\operatorname{coker}(\mathcal{D})]$. For more details on almost complex quantization and on stable complex quantization see [CKT].

Cohomological invariants of a Hamiltonian space. These are defined in an analogous way to cohomological invariants of a symplectic manifold, only that we replace the relevant cohomology classes by their equivariant counterparts, as we now explain.

A T-equivariant differential form is a polynomial map from the Lie algebra $\mathfrak{t}$ to the space of T-invariant differential forms on M. If we view the moment map Φ as a linear map from $\mathfrak{t}$ to $C^\infty(M) = \Omega^0(M)$ then $\tilde{\omega} := \omega + \Phi$ is an equivariant differential form. One defines an exterior derivative on equivariant differential forms. The cohomology of the resulting differential complex is the equivariant cohomology, $H_T^*(M,\mathbb{R})$.

The form $\tilde{\omega} = \omega + \Phi$ is equivariantly closed so we can take its cohomology class $[\tilde{\omega}] \in H_T^2(M)$. An equivariant almost complex (or stable complex) structure gives us equivariant Chern classes, $\tilde{c}_i \in H_T^{2i}(M)$. *Cohomological invariants* of a Hamiltonian space are invariants which only depend on the equivariant characteristic numbers, i.e., on the integrals $\int_M P(\tilde{\omega}, \{\tilde{c}_i\})$ where P is a polynomial. Note that equivariant characteristic numbers are not numbers; they are functions on $\mathfrak{t}$.

Again, quantization provides a nice example. In the presence of a T-action which preserves all the structures in sight, the quantization of a symplectic manifold becomes a *virtual representation*. As such, it is determined by its character. The character is a function $\chi : \mathfrak{t} \longrightarrow \mathbb{C}$ and is given by $\chi = \int_M \exp(\tilde{\omega})\,\mathrm{Todd}(\{\tilde{c}_i\})$ on a neighborhood of $0 \in \mathfrak{t}$. This formula is obtained from the Atiyah-Segal-Singer index theorem combined with the localization theorem in equivariant cohomology.

Cohomological invariants are preserved under cobordisms. This is because, by Stokes's theorem, the characteristic numbers of a boundary are all zero.

Theorem 1. *Let M be a compact manifold with a circle action with isolated fixed points. Then M is cobordant, via orbifolds, to the disjoint union of twisted projective spaces;*

$$M \cong \bigcup_{p \in M^T} \overline{T_pM}.$$

This union is over the (finite) set of fixed points in M. The space $\overline{T_pM}$ is the twisted projective completion *of the vector space T_pM; we have $\overline{T_pM} = T_pM \sqcup \mathbb{P}(T_pM)$ where $\mathbb{P}(T_pM) :=$(a sphere in T_pM)$/S^1$.*

Proof. Consider the product $M \times \mathbb{C}$ with the anti-diagonal circle action, $a \cdot (m,z) = (a \cdot m, a^{-1}z)$. The fixed points in $M \times \mathbb{C}$ are the points of the form $(p,0)$ where p is a fixed point in M. For every such p let U_p be a small ball around $(p,0)$ in $M \times \mathbb{C}$ (where we identify a neighborhood of p in M with a neighborhood of 0 in T_pM). Let $U_\infty = \{(m,z) \in M \times \mathbb{C} \mid |z| > R\}$ for some large R. The U_p's are neighborhoods of the fixed points in $M \times \mathbb{C}$ and U_∞ is a neighborhood of infinity. Let $\tilde{W}$ be the complement of $\cup_p U_p \cup U_\infty$ in $M \times \mathbb{C}$ and consider the quotient $W = \tilde{W}/S^1$. This is an orbifold. Its boundary components are $\partial U_\infty/S^1 = M$ and $\partial U_p/S^1 = \overline{T_pM}$. □

The cobordism in Theorem 1 can be made to carry further structure: a torus action (which contains the circle action), a closed 2-form, a moment map, a stable complex structure. Theorem 1 is also true for non-isolated fixed points, when one replaces the T_pM's by the normal bundles to the components of the fixed point set; see [GGK].

Remark. In Theorem 1, although we start with a smooth manifold M we end up with orbifolds $\overline{T_pM}$. If we start with isolated fixed points then we may end up with non-isolated fixed points.

Fixed point data. Let (M, ω, Φ) be a Hamiltonian T-space with isolated fixed points. Its *fixed point data* is: for every fixed point $p \in M^T$, the moment map value $\Phi(p) \in \mathfrak{t}^*$, and the isotropy weights by which the torus T acts on the tangent space T_pM. (The isotropy weights are elements of the integral weight lattice in $\mathfrak{t}^*$.) Two spaces are said to have the *same fixed point data* if there exists a bijection between their sets of fixed points such that corresponding fixed points have the same isotropy weights and the same moment map value.

Theorem 1 implies that if two spaces have the same fixed point data then they are cobordant. Conversely, if two spaces are cobordant then they have the same fixed point data: one can recover the fixed point data from the equivariant characteristic numbers; in other words, the fixed point data is a cohomological invariant; see [GGK].

Summary. For two compact Hamiltonian T-spaces with isolated fixed points the following equivalence relations are equivalent:

1. cobordant
2. same cohomological invariants
3. same fixed point data.

Cobordism versus isomorphism. An *isomorphism* between two Hamiltonian T-spaces, (M_i, ω_i, Φ_i), i=1,2, is a diffeomorphism $F : M_1 \longrightarrow M_2$ such that $F^*\omega_2 = \omega_1$ and $F^*\Phi_2 = \Phi_1$ (consequently F is T-equivariant).

Example 1. Two cobordant spaces need not be isomorphic, as the following example shows. Let $T = S^1 \times S^1$. Figure 1 shows two polygons in $\mathfrak{t}^* = \mathbb{R}^2$ which are the moment images of two Delzant spaces. The dots denote the weight lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ and the integer labels will be explained soon. The moment map Φ maps the two spaces onto the two polygons. The preimage of a vertex is a fixed point. The preimage of an interior point in a polygon is a free T-orbit in the corresponding space. If a point lies on an edge with

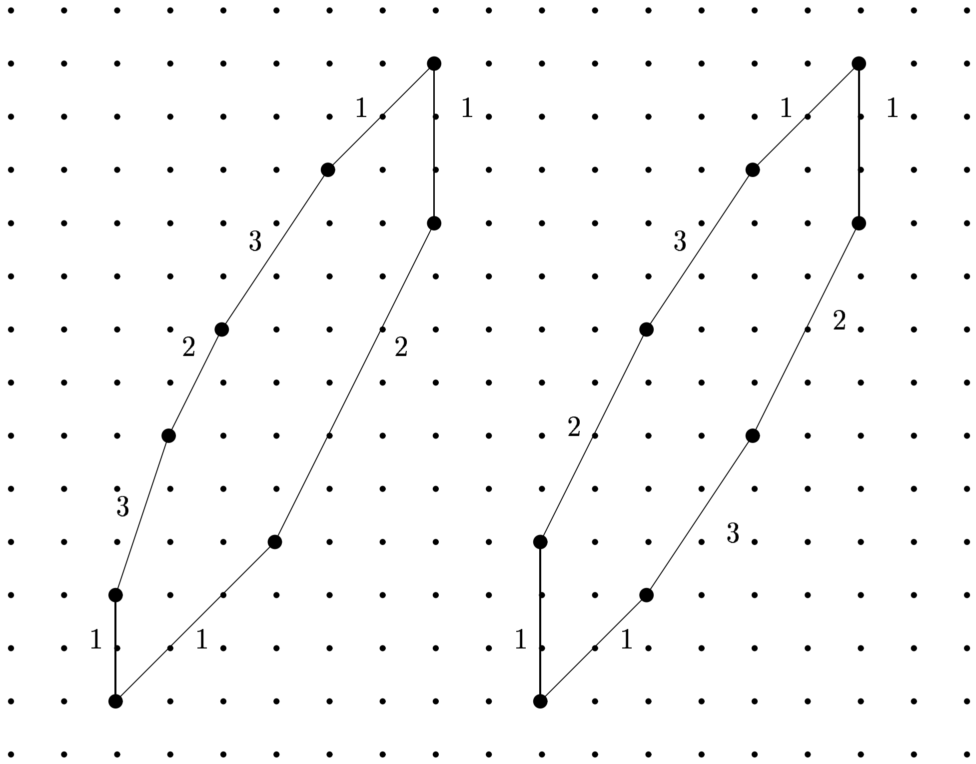

FIGURE 1. Cobordant but non-isomorphic Hamiltonian S^1-spaces

moment map value	isotropy weights
0	1,1
2	3,−1
3	2,−1
5	2,−3
7	3,−2
9	1,−2
10	1,−3
12	−1,−1

FIGURE 2. The fixed point data of the spaces in Figure 1

slope k/b but not on a vertex then the preimage of that point is an orbit whose stabilizer is the subgroup $\{(\lambda^k, \lambda^{-b}) \mid |\lambda| = 1\}$ of $S^1 \times S^1$. These facts follow immediately from the local normal form for Hamiltonian torus actions and from the connectedness of the level sets of the moment map, see [GS].

Now let us restrict the torus action to the action of the subcircle $\{e\} \times S^1$, but still consider the T-moment map, Φ. The fixed points are still the preimages of the vertices. If $\Phi(p)$ lies on an edge of a polygon, but is not a vertex, then the stabilizer of p is the cyclic group with k elements, where the integer k is as labled in Figure 1. One can easily deduce this from the above remarks by computing the slopes of the edges. These integers determine the S^1-isotropy weights at the fixed points: if the edge coming up from a vertex is labeled m and the edge coming down is labeled n then the preimage of the vertex is a fixed point with isotropy weights m and $-n$; this again follows from the local normal form of Guillemin and Sternberg. Finally, the moment map for the restricted circle action is the map Φ to the polygon composed with the height function $(x, y) \mapsto y$ on the plane. Hence the fixed point data for the circle action can be easily read from the polygon. It is not hard to see that the fixed point data for our two S^1-spaces is the same. This data is listed in Figure 2.

Hence, by Theorem 1, our two S^1-spaces are cobordant. However, they are not isomorphic. They are not even S^1-equivariantly diffeomorphic because their orbit type strata are arranged differently. For example, the space on the left has a stratum with stabilizer $\mathbb{Z}_2$ whose closure meets the closures of two different strata with stabilizer $\mathbb{Z}_3$, and this is not true for the space on the right.

Consequently, cohomological invariants are not enough to distinguish Hamiltonian spaces. (For symplectic manifolds without group actions this was proved by D. McDuff in [MD]; that case is more difficult than ours.)

PART 3 : X-RAYS

Definition. Let (M, ω, Φ) be a Hamiltonian T-space. An orbit type stratum is a connected component of the set $\{p \in M \mid \operatorname{Stab}(p) = H\}$ for some subgroup $H \subseteq T$. Denote by χ the set of all orbit type strata. The *x-ray* of (M, ω, Φ) (over $\mathfrak{t}^*$) consists of the indexing set χ together with the following data.

1. For every $X \in \chi$, the set $\Phi(X) \subseteq \mathfrak{t}^*$.
2. For every $X \in \chi$, its stabilizer group $H \subseteq T$.
3. For every $X \in \chi$, its set of normal characters $\nu = \{\rho_1, \dots, \rho_k\}$, where $2k = \operatorname{codim} X$ and $\rho_i : H \longrightarrow S^1$ are defined below.
4. For every $X, Y \in \chi$, the information: whether X is contained in the closure of Y.

The normal characters are the homomorphisms $\rho_i : H \longrightarrow S^1$ such that the action of H on a fiber of the normal bundle of X in M is isomorphic to the representation of H on $\mathbb{C}^k$ given by $a \cdot (z_1, \dots, z_k) = \rho_1(a) z_1, \dots, \rho_k(a) z_k$. Here we take a complex structure on the normal bundle which is compatible with the symplectic structure; the ρ_i's are then determined up to a permutation. Notice that we do not keep track of the internal geometry of the strata X.

Isomorphism and local isomorphism of x-rays. The x-rays of two Hamiltonian T-spaces spaces are *isomorphic* if there exists a bijection between their indexing sets χ which preserves the data (1)–(4).

The x-rays of two Hamiltonian T-spaces (M_i, ω_i, Φ_i), $i = 1, 2$, are *locally isomorphic* if every point in $\mathfrak{t}^*$ has a neighborhood U such that the x-rays of the Hamiltonian T-spaces $(\Phi_i^{-1}(U), \omega_i, \Phi_i)$ are isomorphic. This is a condition on the x-rays of the spaces and not on the spaces themselves.

Example. The S^1-spaces illustrated in Figure 1 have different x-rays but the same local x-ray (over $\mathfrak{t}^* = \mathbb{R}$).

Local isomorphism of Hamiltonian spaces. We say that two Hamiltonian T-spaces (M_i, ω_i, Φ_i), $i = 1, 2$, are *locally isomorphic* if every point in $\mathfrak{t}^*$ has a neighborhood U over which the Hamiltonian T-spaces $(\Phi_i^{-1}(U), \omega_i, \Phi_i)$, $i = 1, 2$, are isomorphic.

Spaces of deficiency 1. We expect to be able to get a good understanding of compact spaces of deficiency 1, i.e., spaces for which the dimension of the torus is one less than half the dimension of the manifold. This is because the reduced spaces $M//T$ are then two dimensional symplectic surfaces. These are well understood; a closed symplectic surface is determined up to isomorphism by its genus and its total area. The reduced surfaces $M//T$ may have orbifold singularities, but these are accounted for by the torus action.

For example, compact Hamiltonian S^1-spaces of dimension 4 have been completely classified in [K] (also see [AH, A]).

More precisely, Tolman and I expect to prove in [KT] that for a compact connected Hamiltonian T-space of deficiency 1 with isolated fixed points, the local x-ray determines the space up to local isomorphism; moreover, the local x-ray is determined by the fixed point data. Consequently, we expect that for such spaces the following equivalence relations would be equivalent:

1. locally isomorphic
2. same local x-ray
3. same fixed point data
4. cobordant
5. same cohomological invariants.

The equivalence of (3), (4) and (5) was discussed in Part 2 of these notes.

Global isomorphism of Hamiltonian spaces. Unfortunately, the x-ray of a space does not determine the space. In [KT] we will describe two compact, connected Hamiltonian T-spaces of deficiency 1 and with isolated fixed points, which are non-isomorphic but which have isomorphic x-rays.

To every Hamiltonian T-spaces (M, ω, Φ) one can associate a sheaf $\mathcal{S}$ of "local isomorphisms" over $\mathfrak{t}^*$, such that the set of Hamiltonian spaces which are locally isomorphic to (M, ω, Φ) is parametrized by the first cohomology, $H^1(\mathcal{S})$. Fortunately, in some cases we can compute $H^1(\mathcal{S})$; see [KT]. For example, the S^1-space illustrated in Figure 1 on the left has $H^1(\mathcal{S}) = \mathbb{Z}_2$. The space on the right is locally isomorphic to it but not globally isomorphic to it. These two spaces correspond to the two elements of $H^1(\mathcal{S})$. Details will appear in [KT].

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INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, GIV'AT RAM, JERUSALEM 91904, ISRAEL

E-mail address: `karshon@math.huji.ac.il`