CHAPTER 5

Reduction and applications

We begin this chapter by describing "symplectic" reduction of closed but not necessarily non-degenerate two-forms. We then generalize this procedure to an abstract moment map, while keeping track of various additional structures. This treatment is partially taken from [**CKT**], where we carefully reduced orientations, almost complex structures, stable complex structures, and Spin^c structures. (In this book we postpone Spin^c reduction to Section 3.4 of Appendix D.) We then prove the Duistermaat–Heckman theorem about the variation of the reduced symplectic form. Next, we explain the reduction of Kähler structures and consider an important special case—toric varieties, as reductions of $\mathbb{C}^d$. We prove that cobordism commutes with reduction, i.e., cobordant spaces have cobordant reductions. Together with the linearization theorem of Chapter 4, this implies that, for a Hamiltonian torus action with isolated fixed points, the reduced space is cobordant to a disjoint union of (stable complex) toric varieties. This important fact was observed by Shaun Martin in his study of the cohomology ring of reduced spaces [**Mart1**]. We use this fact in Chapter 8, in our cobordism proof of "quantization commutes with reduction". Finally, we give a topological interpretation of the Jeffrey–Kirwan localization, which was one of the motivations for the development of this cobordism theory.

1. (Pre-)symplectic reduction

Let (M, ω, Φ) be a Hamiltonian G-space for a torus G. Recall that by definition the moment map Φ is G-invariant. Hence, G acts on each level set of the moment map Φ. The *reduced space* is the quotient

$$M_\alpha = Z/G, \qquad Z = \Phi^{-1}(\alpha).$$

It is useful to keep in mind the inclusion-quotient diagram

$$(5.1) \qquad \begin{array}{ccc} Z & \overset{i}{\hookrightarrow} & M \\ \pi \downarrow & & \\ M_\alpha & & \end{array}$$

More generally, we may consider the reduced space $M_\alpha = \Phi^{-1}(\alpha)/G$ whenever G is a Lie group acting properly on M and the value α is in the subspace $(\mathfrak{g}^*)^G$ which is fixed under the coadjoint action (e.g., $\alpha = 0$), so that G acts on $\Phi^{-1}(\alpha)$. (If in the non-abelian case α is not preserved by the coadjoint action, the reduction procedure requires some modifications. For example, the G-action on Z should be replaced by the G_α-action.)

Suppose that α is a regular value for $\mathfrak{g}^*$. Then the level set $\Phi^{-1}(\alpha)$ is a manifold of dimension $\dim M - \dim G$, by the implicit function theorem. Also, the

G-action on $\Phi^{-1}(\alpha)$ is locally free. This means that the stabilizers are discrete, or, equivalently, that the infinitesimal stabilizer

$$\mathfrak{g}_p = \{\xi \in \mathfrak{g} \mid \xi_M(p) = 0\}$$

(see Section 1.5 of Appendix B) is trivial for all p. Indeed, since α is a regular value, $d\Phi^\xi_p \neq 0$ for any $\xi \in \mathfrak{g}$. Then, since

$$d\Phi^\xi|_p = \iota(\xi_M)\omega|_p,$$

we see that $\xi_M(p) \neq 0$ for any $\xi \in \mathfrak{g}$ which exactly means that the action is locally free, i.e., $\mathfrak{g}_p = 0$.

The quotient $M_\alpha = Z/G$ is then an orbifold and the map $Z \to M_\alpha$ is a principal G-orbi-bundle. (See Section 3.1 of Appendix B.) In the special case that G acts on Z *freely*, M_α is a manifold and $Z \to M_\alpha$ is a principal G-bundle.

In particular, we have

$$\dim M_\alpha = \dim M - 2\dim G \quad \text{ if } \alpha \text{ is regular.}$$

(Pre-)symplectic reduction asserts that the two-form ω on M descends to a two-form ω_α on the quotient M_α. Originally symplectic reduction was introduced for symplectic forms (see, e.g., [**MW**]); hence the name. However, as was later observed, it is sufficient to assume that ω is closed and G-invariant:

THEOREM 5.1 (Symplectic reduction). *Let (M, ω, Φ) be a Hamiltonian G-space. Suppose that $\alpha \in (\mathfrak{g}^*)^G$ is a regular value for Φ; or, more generally, that the level set $\Phi^{-1}(\alpha)$ is a manifold and G acts on it locally freely. Then there exists a unique closed two-form ω_α on M_α such that $\pi^*\omega_\alpha = i^*\omega$, where $\pi\colon Z \to M_\alpha$ is the quotient map and $i\colon Z \to M$ is the inclusion map. The reduced form ω_α is non-degenerate on M_α if and only if the form ω is non-degenerate on M at the points of $\Phi^{-1}(\alpha)$.*

If a Lie group G acts properly and locally freely on a manifold Z, a differential form β on Z is the pullback of some form on the orbifold G/Z if and only if β is *basic*, meaning that β satisfies the following two conditions. First, β is G-invariant. Second, β is horizontal, meaning that $\iota(\xi_Z)\beta = 0$ for all the vector fields ξ_Z, $\xi \in \mathfrak{g}$, that generate the action.

PROOF OF THEOREM 5.1. The two-form $i^*\omega$ is G-invariant because ω is G-invariant; it is horizontal because $\iota(\xi_M)i^*\omega = d\Phi^\xi \circ i$, which vanishes because Φ is constant on $\Phi^{-1}(\alpha)$. Hence, $i^*\omega$ is basic, and there exists an ω_α such that $\pi^*\omega_\alpha = i^*\omega$.

To prove the non-degeneracy assertion of the theorem, first note that the null space of ω in Z is equal to the sum of the null space of ω in M and the tangent space to the orbit. In other words, for $m \in Z$,

$$\begin{aligned}&\{u \in T_mZ \mid \omega(u,v) = 0 \text{ for all } v \in T_mZ\}\\ &= \{u \in T_mM \mid \omega(u,v) = 0 \text{ for all } v \in T_mM\} + \{\xi_M(m) \mid \xi \in \mathfrak{g}\}.\end{aligned}$$

We leave the proof of this fact to the reader as an exercise. Furthermore, a vector u belongs to the null space of ω in Z if and only if its image π_*u in TM_α belongs to the null space of ω_α in M_α. This implies that ω is non-degenerate at the points of Z if and only ω_α is non-degenerate. ☐

EXAMPLE 5.2. Let S^1 act on $\mathbb{C}^{n+1}$ by scalar multiplication. The moment map for the standard symplectic form is $\Phi(z) = \frac{1}{2}\sum_{i=0}^n |z_i|^2$. Its level set $Z = \Phi^{-1}(\alpha)$

is the $(2n+1)$-sphere in $\mathbb{C}^{n+1}$ of radius $\sqrt{2\alpha}$. The quotient $M_\alpha = Z/S^1$ is $\mathbb{CP}^n$. The reduced symplectic form ω_α is the *Fubini–Study* form on $\mathbb{CP}^n$.

EXAMPLE 5.3. Take the phase space of a system of N particles, $\mathbb{R}^{6N}$, with $\mathbb{R}^3$ acting by translations, as described in Example 2.3. The reduced space M_α is obtained by fixing the total linear momentum and ignoring simultaneous translations of the N particles. Hence, M_α describes the system *relative to the center of mass.*

Symplectic reduction provides a notion of a quotient in the symplectic category. Namely, suppose that we have a symplectic manifold M with an action of a group G. The naive set theoretic quotient M/G is not naturally symplectic; it might even be odd-dimensional. Instead, one defines the symplectic quotient to be the reduced space $\Phi^{-1}(0)/G$. This is denoted $M//G$. In spite of its ambiguity (if the moment map is allowed to shift), this definition works well for many purposes.

If α is a singular value of the moment map Φ, the reduced space $M_\alpha = \Phi^{-1}(\alpha)$ is a *stratified space*, and the two-form ω still reduces to a two-form ω_α on M_α, in an appropriate sense. See [**ACG, SL**] and references therein.

2. Reduction for abstract moment maps

Let G be a torus. Recall that an abstract moment map on a G-manifold M is a smooth invariant map

$$\Psi\colon M \to \mathfrak{g}^*$$

such that for any subgroup H of G, the H-component $\Psi^H\colon M \to \mathfrak{h}^*$ is locally constant on the H-fixed point set M^H. (See Chapter 3.)

As for ordinary moment maps, we define the reduced space

$$M_\alpha = Z/G, \qquad Z = \Phi^{-1}(\alpha),$$

and consider the inclusion-quotient diagram

$$\begin{array}{ccc} Z & \overset{i}{\hookrightarrow} & M \\ \pi\downarrow & & \\ M_\alpha & & \end{array}$$

2.1. Regular reduced spaces. If (M,ω) is a symplectic manifold with a torus action and Φ is a moment map, the torus acts locally freely on the regular level sets of Φ. The same holds for *abstract moment maps*; the proof is only slightly harder:

LEMMA 5.4. *Let a torus G act on a manifold M with an abstract moment map Ψ. If α is a regular value of Ψ, the G-action on the level set $\Psi^{-1}(\alpha)$ is locally free.*

PROOF. It is enough to prove that every point whose stabilizer is not discrete is a critical point for Ψ.

First, let us assume that G is a circle. Let F be a component of its fixed point set. Since Ψ is constant on F, the restriction of $d\Psi$ to the tangent bundle of F is zero. Since Ψ is invariant under the circle action, the restriction of $d\Psi$ to the normal bundle of F is zero. Indeed, at each $p \in F$ the normal space to F is a vector space with a circle action that only fixes the origin. On such a vector space, for every invariant function, the origin is a critical point.

Hence, every fixed point for the circle action is a critical point for the abstract moment map.

Now, let G be a torus of any dimension. We need to show that any point $m \in M$ whose stabilizer has positive dimension is a critical point for Ψ. Let H be a circle subgroup of the stabilizer of m. The previous paragraph, applied to the action of H, implies that m is a critical point for the H-component of Ψ, hence for Ψ. □

By Lemma 5.4, if α is a regular value of Ψ, the *reduced space* M_α is an orbifold. If Φ is proper, M_α is compact.

2.2. Orientations on reduced spaces. Let M be a manifold with a proper action of a torus G and an abstract moment map

$$\Psi\colon M \to \mathfrak{g}^*.$$

Let $\alpha \in \mathfrak{g}^*$ be a regular value for Ψ. Recall that if G is not a torus, we assume for the sake of simplicity that $\alpha = 0$ or, more generally, that α is G-invariant. As above, consider the reduced space

$$M_\alpha = Z/G \quad , \quad Z = \Phi^{-1}(\alpha).$$

In this section we show that an orientation on M naturally induces an orientation on M_α and that an equivariant stable complex structure on M naturally induces a stable complex structure on M_α.

We have an exact sequence of vector bundles over Z,

$$0 \to TZ \to TM|_Z \to NZ \to 0, \tag{5.2}$$

which follows immediately from the definition of the normal bundle. By the implicit function theorem, the normal bundle is trivial:

$$NZ = Z \times \mathfrak{g}^* \tag{5.3}$$

and an isomorphism $T_pM/T_pZ \cong \mathfrak{g}^*$ is given by the map $d\Psi$. The tangent spaces to the G-orbits form a sub-bundle of TZ which we denote $T\mathcal{O}$. This bundle is also trivial:

$$T\mathcal{O} = Z \times \mathfrak{g}, \tag{5.4}$$

where an isomorphism $\mathfrak{g} \cong T_p(G \cdot p)$ is obtained by sending ξ to $\xi_M(p)$. By the definition of the reduced space, the sequence

$$0 \to T\mathcal{O} \to TZ \to \pi^*TM_\alpha \to 0 \tag{5.5}$$

is exact. From (5.2)–(5.5), we obtain an isomorphism of G-equivariant vector bundles,

$$TM|_Z \cong \pi^*TM_\alpha \oplus \mathfrak{g} \oplus \mathfrak{g}^*. \tag{5.6}$$

This isomorphism is not canonical: it depends on the choices of splittings of the exact sequences. However, different choices lead to equivariantly homotopic isomorphisms (5.6). Indeed, the difference between two equivariant splittings of (5.2) is an invariant section of the vector bundle over Z whose fiber over $p \in Z$ is formed by linear maps from NZ_p to T_pZ. The space of such sections is contractible. A similar argument applies to the splittings of (5.5).

The vector space $\mathfrak{g}\oplus\mathfrak{g}^*$ is naturally oriented: a basis for $\mathfrak{g}$ together with the dual basis in $\mathfrak{g}^*$ determine a canonical (i.e., independent of the basis of $\mathfrak{g}$) orientation of $\mathfrak{g} \oplus \mathfrak{g}^*$. Then by (5.6), an orientation on M induces an orientation on M_α.

2.3. Stable complex reduction. In this section we will show that a G-equivariant stable complex structure J on M induces a stable complex structure J_α on the reduced space M_α.

By definition, J is an invariant complex structure on the fibers of a Whitney sum $TM \oplus \mathbb{R}^k$, where $\mathbb{R}^k$ is equipped with the trivial G-action. We use two equivalence relations on such structures. Suppose that J_0 and J_1 are complex structures on $TM \oplus \mathbb{R}^{k_0}$ and $TM \oplus \mathbb{R}^{k_1}$. These stable complex structures are *bundle equivalent* if $TM \oplus \mathbb{R}^{k_0}$ and $TM \oplus \mathbb{R}^{k_1}$ become equivariantly *isomorphic* complex vector bundles after adding some trivial complex vector bundles. The structures J_0 and J_1 are *homotopic* if the complex structures on $TM \oplus \mathbb{R}^{k_0}$ and $TM \oplus \mathbb{R}^{k_1}$ become equivariantly *homotopic* after adding some trivial complex vector bundles. Homotopic structures are always bundle equivalent, but not vice versa. We refer the reader to Section 1 of Appendix D for a detailed discussion of these notions.

From (5.6) we obtain an isomorphism

$$\pi^* TM_\alpha \oplus \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathbb{R}^k \cong T_Z M \oplus \mathbb{R}^k. \tag{5.7}$$

The structure J transports through this isomorphism to a G-invariant complex structure on the fibers of $\pi^* TM_\alpha \oplus \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathbb{R}^k$. This complex structure descends to a complex structure on the fibers of $TM_\alpha \oplus \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathbb{R}^k$. We identify

$$\mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathbb{R}^k \cong \mathbb{R}^s \tag{5.8}$$

where $s = 2\dim G + k$, and obtain a complex structure on the fibers of

$$TM_\alpha \oplus \mathbb{R}^s,$$

i.e., a stable complex structure J_α on M_α.

The isomorphism (5.7) is canonical up to homotopy. We insist that the linear isomorphism (5.8) respect the orientations, hence, it is also determined uniquely up to homotopy. Consequently, the reduced stable complex structure J_α is canonical up to homotopy.

Homotopic equivariant stable complex structures on M reduce to homotopic stable complex structures on M_α, and bundle equivalent equivariant stable complex structures on M reduce to bundle equivalent stable complex structures on M_α.

2.4. Relation with symplectic reduction. We recall that an (invariant) symplectic structure naturally determines an (invariant) almost complex structure, unique up to homotopy, and hence an (equivariant) stable complex structure, unique up to homotopy. (See Section 1.3 of Appendix D.) In this case, the two notions of reductions are consistent with each other. Namely, let (M, ω) be a symplectic manifold and Ψ a genuine moment map for a G-action on M. The reduced space M_α is again a symplectic manifold, with symplectic form ω_α. Let J be an equivariant stable complex structure on M that is associated with ω, and let J_0 be a stable complex structure on M_α that is associated with ω_α. We claim that J_0 is homotopic to the stable complex structure J_α that is obtained from J by reduction.

Choose a basis of $\mathfrak{g}$ and the dual basis in $\mathfrak{g}^*$. Then the isomorphism (5.6) takes the form

$$T_Z M \cong \pi^* TM_\alpha \oplus \mathbb{R}^{2\dim G}. \tag{5.9}$$

We choose this isomorphism in such a way that ω descends to the form $\omega_\alpha \oplus \sigma$, where σ is the standard symplectic form on $\mathbb{R}^{2\dim G}$. (The fact that this is possible can be seen, e.g., from the proof of the symplectic reduction theorem using local coordinates. We leave the details to the reader as an exercise.)

Let J_{std} be the standard complex structure on $\mathbb{R}^{2\dim G} = \mathbb{C}^{\dim G}$. This structure is compatible with the standard symplectic form. Let J'_α be an almost complex structure on M_α that is compatible with ω_α. Then $\pi^* J'_\alpha \oplus J_{\text{std}}$ transports via (5.9) to an invariant almost complex structure J' on M that is compatible with ω. From the definition of reduction, $(J')_\alpha = J'_\alpha$. Because J' is compatible with ω, J' is homotopic to J, and hence J'_α is homotopic to J_α, as required.

In contrast with the symplectic case, in general an almost complex structure on M need not descend to an almost complex structure on M_α. In fact, starting with an almost complex structure J on M, let us reduce J to a stable complex structure J_α on M_α with respect to a moment map that is unrelated to J. Then J_α may fail to be homotopic, or even bundle equivalent, to an almost complex structure on M_α. This is the main reason that stable complex structures arise in the context of this book.

Example 5.5. Consider $M = \mathbb{C}^n$ with the diagonal action of $G = S^1$ and moment map $\Psi(z) = \sum |z_j|^2$. The reduction at $\Psi = 1$ is the complex projective space $\mathbb{CP}^{n-1}$. Let J be the non-standard complex structure on $\mathbb{C}^n$ whose complex coordinates are $(\overline{z}_1, \dots, \overline{z}_r, z_{r+1}, \dots, z_n)$ for $0 \leq r \leq n$. Note that J is not compatible with the standard symplectic structure on $\mathbb{C}^n$ for which Ψ is a genuine moment map.

We claim that *unless $r = 0$, or $r = n$ and is odd, the reduced stable complex structure J_α is not induced by any almost complex structure on* $\mathbb{CP}^{n-1}$. Indeed, an easy calculation shows that the total Chern class of $(T\mathbb{CP}^{n-1}, J_\alpha)$ is $(1-u)^r(1+u)^{n-r}$, where u is a suitably chosen generator of $H^2(\mathbb{CP}^{n-1};\mathbb{Z})$. The integral of the top Chern class c_n over $\mathbb{CP}^{n-1}$ is $(-1)^{r-1}r + (-1)^r(n-r)$. Recall that for any complex vector bundle the top Chern class is equal to the Euler class; see, e.g., [**MiSt**]. Therefore, if J_α is induced by an almost complex structure, this Chern number must be equal to the Euler number $\chi(\mathbb{CP}^{n-1}) = n$. In other words, we must have

$$(-1)^{r-1}r + (-1)^r(n-r) = n,$$

which can happen only when $r = 0$ or when $r = n$ and is odd.

2.5. Reduction of cohomology. A two-form on M does not naturally descend to M_α. However, an equivariant cohomology class does. Here we just outline the construction of reduction in cohomology; for more details and relevant definitions we refer the reader to Section 2 of Appendix C.

If the G-action on Z is free, the natural pullback map

$$\pi^* \colon H^*(Z/G) \to H^*_G(Z)$$

is an isomorphism (for cohomology with real coefficients). The inverse map

$$H^*_G(Z) \to H^*(Z/G)$$

is called the *Cartan map*. To each equivariant cohomology class c on M we associate an ordinary cohomology class c_α on M_α by applying the Cartan map to the restriction of c to Z:

$$\pi^* c_\alpha = i^* c. \tag{5.10}$$

The map

$$c \mapsto c_\alpha \tag{5.11}$$

is the famous *Kirwan map*; it is a ring homomorphism from the equivariant cohomology of M to the ordinary cohomology of the reduced space M_α. Kirwan's *surjectivity theorem* asserts that if Ψ is a moment map for a symplectic form on a compact manifold, then the map (5.11) is onto.

The *reduction* of the triple (M, Ψ, c) is defined to be the pair (M_α, c_α).

3. The Duistermaat–Heckman theorem

Let (M, ω, Φ) be a Hamiltonian G-space with a proper moment map

$$\Phi\colon M \to \mathfrak{g}^*.$$

Denote by $\mathfrak{g}^*_{\mathrm{reg}}$ the subset of $\mathfrak{g}^*$ consisting of the regular values of Φ. The set $\mathfrak{g}^*_{\mathrm{reg}}$ is open (because Φ is proper) and dense (by Sard's theorem). The theorem of Duistermaat and Heckman is concerned with the following question:

How does the pair $(M_\alpha, \omega_\alpha)$ vary as α varies in $\mathfrak{g}^*_{\mathrm{reg}}$?

The answer depends on the topology of the orbifold fibration

$$\pi\colon Z \to M_\alpha$$

and, in particular, on the *curvature class* of this fibration. Let us first recall the definition of this class.

3.1. Variation of the reduced form. From now on, we assume that G is a torus. Let α be a regular value of Φ. Over a neighborhood of α, the moment map Φ is a proper submersion. Hence, Φ is a fibration (by *Ehresmann's lemma*, which asserts that a proper submersion is locally a projection). This remains true equivariantly: the G-spaces $\Phi^{-1}(\alpha+\varepsilon)$, for $\varepsilon \in \mathfrak{g}^*$ sufficiently small, are equivariantly diffeomorphic to each other. Therefore, the reduced forms $\omega_{\alpha+\varepsilon}$ can be considered as all living on M_α. Let $[\omega_{\alpha+\varepsilon}] \in H^2(M_\alpha)$ denote the cohomology class of $\omega_{\alpha+\varepsilon}$.

Theorem 5.6 ([**DH1**]). *For $\varepsilon \in \mathfrak{g}^*$ near 0,*

$$[\omega_{\alpha+\varepsilon}] = [\omega_\alpha] - \langle \varepsilon, c\rangle$$

where $c \in H^2(M_\alpha) \otimes \mathfrak{g}$ is the curvature class of the fibration $\pi\colon Z \to M_\alpha$.

Proof. Let W be a small neighborhood of α, so that we may identify its preimage as $\Phi^{-1}(W) = Z \times W$. Let τ be a ($\mathfrak{g}$-valued) connection one-form on the principal G-orbi-bundle

$$p\colon \Phi^{-1}(W) \to \Phi^{-1}(W)/G. \tag{5.12}$$

The curvature is a $\mathfrak{g}$-valued closed two-form F on $(Z/G)\times W$ such that $p^*F = -d\tau$. Its restriction, $F_{\alpha+\epsilon}$ to $(Z/G) \times \{\alpha+\epsilon\}$, represents the curvature class c of the fibration $Z \to Z/G$, for each $\alpha + \epsilon \in W$. We need to show that the cohomology class

$$[\omega_{\alpha+\epsilon}] - \langle \epsilon, c\rangle = [\omega_{\alpha+\epsilon} - \langle \epsilon, F_{\alpha+\epsilon}\rangle]$$

on Z/G is independent of ϵ. This would follow from homotopy invariance if we show that the forms

$$\omega_{\alpha+\epsilon} - \langle \epsilon, F_{\alpha+\epsilon}\rangle$$

are the restriction to the fibers of a *closed* form on $(Z/G)\times W$. Indeed, these forms are the restrictions to the fibers of the form which pulls back to the closed form

$$\omega + d\langle \Phi - \alpha, \tau\rangle$$

on $Z \times W$. □

3.2. Volume of reduced spaces. Consider the regular reduced spaces

$$M_\alpha = \Phi^{-1}(\alpha)/G, \qquad \alpha \in \mathfrak{g}^*_{\mathrm{reg}}.$$

The *Duistermaat–Heckman function* associates to every regular value α the Liouville volume of the corresponding reduced space

$$\varphi(\alpha) = \mathrm{volume}(M_\alpha) = \int_{M_\alpha} \frac{\omega_\alpha^d}{d!},$$

where $d = \frac{1}{2}\dim M - \dim G$ is half the dimension of the reduced space, ω_α is the reduced two-form, and we integrate with respect to the reduced orientation. An important aspect of the Duistermaat–Heckman theorem is the following result:

THEOREM 5.7. *The Duistermaat–Heckman function is a polynomial in α of degree at most $d = \frac{1}{2}\dim M - \dim G$ on each connected component of $\mathfrak{g}^*_{\mathrm{reg}}$.*

PROOF. Fix a regular value α in $\Phi(M)$ and consider values of $\varepsilon \in \mathfrak{g}^*$ near 0. By the Duistermaat–Heckman theorem (see Section 3.1), identifying M_α with $M_{\alpha+\varepsilon}$, we have

$$\mathrm{vol}(M_{\alpha+\varepsilon}) = \int_{M_\alpha} \frac{\omega_{\alpha+\epsilon}^d}{d!} = \int_{M_\alpha} \frac{1}{d!}(\omega_\alpha - \langle \varepsilon, c\rangle)^d, \tag{5.13}$$

where

$$d = \frac{1}{2}\dim M_\alpha = \frac{1}{2}\dim M - \dim G$$

and c is the curvature class of the fibration $Z \to Z/G$. This is a polynomial in ε of degree at most d. □

Recall that the Duistermaat–Heckman measure DH_M on $\mathfrak{g}^*$ is the push-forward via Φ of the Liouville measure.

THEOREM 5.8. *Assume that the action is effective. Then the Duistermaat–Heckman measure can be written as*

$$\mathrm{DH}_M = (2\pi)^{\dim G}\varphi(x)|dx|,$$

where $|dx|$ denotes the Lebesgue measure on $\mathfrak{g}^*$ and $\varphi(x)$ is the Duistermaat–Heckman function.

REMARK 5.9. The Lebesgue measure on $\mathfrak{g}^*$ is normalized in such a way that a fundamental chamber for the lattice $\mathbb{Z}_G^* = \mathrm{Hom}(\mathbb{Z}_G, 2\pi\mathbb{Z})$ in $\mathfrak{g}^*$ has total volume 1.

EXAMPLE 5.10. Let a torus G act on a compact symplectic manifold (M, ω) with a moment map $\Phi\colon M \to \mathfrak{g}^*$. Let us assume that

$$\dim G = \frac{1}{2}\dim M, \tag{5.14}$$

and also that the action is effective. Then (M, ω, Φ) is called a *Delzant space.* (See Section 2 of Chapter 2.) By the convexity theorem, the image $\Delta = \Phi(M)$ is a convex polytope. The dimension assumption (5.14) implies that the regular reduced spaces M_α are zero-dimensional, and hence finite. Because the reduced spaces are connected, they are single points, and their Liouville volumes are all equal to one. The corresponding Duistermaat–Heckman measure is $(2\pi)^{\dim G}|dx|$ on Δ and zero outside Δ, by Theorem 5.8. In particular,

$$\mathrm{vol}(M) = (2\pi)^{\dim G}\mathrm{vol}(\Delta), \tag{5.15}$$

where the volume on the left is the Liouville volume, $\int_M \omega^n/n!$, and the volume on the right is the Euclidean volume of Δ.

In particular, for the standard S^1-action on S^2 with moment map the height function, the Duistermaat–Heckman measure is 2π times the Lebesgue measure on the interval $[-1, 1]$; cf. Example 2.6.

Proof of Theorem 5.8. Consider first a neighborhood of a free orbit in M. On such a neighborhood there exists a coordinate system

$$\theta_1, \ldots, \theta_k, x_1, \ldots, x_k, y_1, \ldots, y_{2d},$$

where $k = \dim G$, $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$, and x_i are coordinates on $\mathfrak{g}^*$, such that the G-action is generated by the vector fields $\frac{\partial}{\partial\theta_1}, \ldots, \frac{\partial}{\partial\theta_k}$ and the moment map is $(x_1, \ldots, x_k)$. The coordinates y_j descend to coordinates on each reduced space M_α. (The fact that the components x_i of the moment map can be taken as coordinates follows from the fact that the points of a free orbit are regular points for the moment map. The existence of the coordinates θ_i and y_i on a level set for the moment map follows from the slice theorem. These coordinates can be extended to neighboring level sets; this follows from Ehresmann's lemma, which guarantees that a proper submersion is locally a projection.)

Hamilton's equation,

$$\iota\left(\frac{\partial}{\partial\theta_i}\right)\omega = dx_i,$$

implies that ω must have the form

$$\begin{aligned}\omega = &\sum d\theta_i \wedge dx_i + \sum e_{ij} dx_i \wedge dx_j \\ &+ \sum f_{ij} dx_j \wedge dy_j + \sum g_{ij} dy_i \wedge dy_j,\end{aligned} \tag{5.16}$$

where the sums are over the appropriate indices i, j, and where the coefficients are functions of the x_i's and y_j's only. The reduced form on $M_\alpha = \Phi^{-1}(\alpha)/G$ is

$$\omega_\alpha = g_{ij}(\alpha, y) dy_i \wedge dy_j.$$

Taking the top wedge product of (5.16), we see that

$$\omega^n/n! = \pm d\theta_1 \wedge \ldots \wedge d\theta_k \wedge dx_1 \wedge \ldots \wedge dx_k \wedge (\omega_\alpha)^d/d!.$$

When we push $\omega^n/n!$ forward, by Fubini's theorem we can first integrate with respect to the θ_i's. This contributes the factor $(2\pi)^k$ to the measure. Then we integrate the form $\omega_\alpha^d/d!$, which leads to the (contribution of the neighborhood to the) Duistermaat–Heckman function, and then integrate with respect to the x_i's over $\mathfrak{g}^*$.

To finish the proof, we argue as follows. Let ϱ_j be an invariant partition of unit, defined on the union of the free orbits in M, such that each ϱ_j is supported in a neighborhood with coordinates as described above. Because the action is free on an open and dense set, the measure defined by integrating the differential form $\sum_j \varrho_j \omega^n/n!$ on this union is equal to the Liouville measure on M. For each j, let f_j be the function which associates to every regular value α of Φ the integral

$\int_{M_\alpha} \varrho_j \omega_\alpha^d/d!$. Then $\sum f_j$ is equal to the Duistermaat–Heckman function on regular values. Finally, a slight modification of the above computation shows that the push-forward to $\mathfrak{g}^*$ of $\varrho_j \omega^n/n!$ is equal to $f_j|dx|$ on $\mathfrak{g}^*$. The theorem follows. □

Theorems 5.8 and 5.7 imply the following theorem of Duistermaat and Heckman:

THEOREM 5.11. *The Duistermaat–Heckman measure on $\mathfrak{g}^*$ has a piecewise polynomial density with respect to the Lebesgue measure.*

The polynomials on $\mathfrak{g}^*$ equal to the density function of the Duistermaat–Heckman measure on the connected components of $\mathfrak{g}^*_{\mathrm{reg}}$ are called the *Duistermaat–Heckman polynomials*. The book [**GLS**] contains explicit recipes for computing these polynomials, examples, pictures, and relations with representation theory.

When the manifold is a coadjoint orbit for a compact Lie group with the action of the maximal torus of the group, the values of the Duistermaat–Heckman polynomials approximate the multiplicities of weights in irreducible representations of the group. This was, in fact, Heckman's original motivation for introducing this push-forward measure; see [**He**].

4. Kähler reduction

Let (M, ω, Φ) be a Hamiltonian G-manifold, equipped with an invariant *Kähler* structure. Then every regular reduced space is a Kähler orbifold. This was shown by Guillemin and Sternberg in [**GS3**]. In this section we sketch their argument.

Recall that a Kähler structure on a symplectic manifold is a compatible complex structure. A complex structure is given by complex local coordinates such that transition functions are holomorphic. This induces an almost complex structure on M, i.e., a fiberwise complex structure, given by a bundle map

$$J\colon TM \to TM$$

such that $J^2 = -$identity. Most almost complex structures are not integrable, i.e., do not come from genuine complex structures. The compatibility condition is that

$$\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot) \tag{5.17}$$

is a (positive definite) Riemannian metric on M.

Suppose that the group G is compact. Let $G_{\mathbb{C}}$ be its complexification. For example, if G is a subgroup of $(S^1)^n$, then its Lie algebra $\mathfrak{g}$ is contained in $i\mathbb{R}^n$, and the complexification of G is the subgroup $G_{\mathbb{C}} = G \cdot \exp(i\mathfrak{g})$ of $(\mathbb{C}^\times)^n$.

Suppose also that the moment map Φ is proper. The G-action naturally extends to a holomorphic $G_{\mathbb{C}}$ action, with the additional generating vector fields $J\zeta_M$, $\zeta \in \mathfrak{g}$.

The definitions of the moment map and the metric (5.17) imply that $d\Phi^\xi(\cdot) = \langle \cdot, -J\xi_M \rangle$, that is, the action of the "non-compact part" of $G_{\mathbb{C}}$ is generated by the gradient vector fields of Φ^ξ, for $\xi \in \mathfrak{g}$.

This, in turn, implies that for any regular level set $Z = \Phi^{-1}(\alpha)$ the map $\mathfrak{g} \times Z \to M$ given by $(\zeta, z) \mapsto \exp(i\zeta) \cdot z$ is open. Therefore,

$$W = G_{\mathbb{C}} \cdot Z$$

is an open subset of M. This is the "stable" subset of M with respect to the G-action and the level $\alpha \in \mathfrak{g}^*$, in the sense of Geometric Invariant Theory (G.I.T.).

The group $G_{\mathbb{C}} \cong \mathfrak{g} \times G$ acts properly and locally freely on $W \cong \mathfrak{g} \times Z$, hence, the G.I.T. quotient

$$M//G := W/G_{\mathbb{C}} \tag{5.18}$$

is a complex orbifold. (See Corollary (B.32).) For every $p \in Z$, the stabilizer group of p in $G_{\mathbb{C}}$ is finite and contained in G. Moreover, every $G_{\mathbb{C}}$-orbit in W intersects Z transversely, in a unique G-orbit. Hence, the natural inclusion map $Z \subset W$ descends to a diffeomorphism

$$Z/G \to W/G_{\mathbb{C}}.$$

In this way, the reduced space Z/G becomes a complex orbifold.

Remark 5.12. The complex structure on the reduced space is compatible with the reduced symplectic structure (see Section 2.4), i.e., the entire Kähler structure descends to the reduced space. However, note that the Riemann metric associated with the Kähler structure on the reduced space is not "induced" from the Riemannian metric on Z. For example, the metric on $\mathbb{C}^n$ is flat, and the metric on the reduced space $\mathbb{CP}^{n-1}$ is not flat.

We refer the reader to [**Kir**] for a broader discussion of the G.I.T. in algebraic geometry and its relation with the symplectic quotient.

Remark 5.13. Kähler reduction is consistent with stable complex reduction. Namely, if J is the stable complex structure associated to the complex manifold M, and J_α is its reduction (as described in Section 2.3), then J_α is (homotopic to) the stable complex structure associated to the complex orbifold Z/G.

5. The complex Delzant construction

5.1. Symplectic reduction of $\mathbb{C}^{\mathbf{d}}$. Let a compact abelian Lie group G act effectively on $\mathbb{C}^d$ with weights $-\alpha_1, \dots, -\alpha_d \in \mathfrak{g}^*$ and moment map

$$\Phi(z) = \frac{1}{2} \sum_{i=1}^{d} |z_i|^2 \alpha_i. \tag{5.19}$$

Assume that the weights are polarized, so that the moment map is proper. (See Proposition 4.15.) Let $\alpha \in \mathfrak{g}^*$ be a regular value for Φ. Consider the level set

$$Z = \Phi^{-1}(\alpha)$$

and the reduced space

$$M_\alpha = Z/G.$$

We also consider the action of $\mathbb{T}^d = (S^1)^d$ on $\mathbb{C}^d$ with moment map

$$J(z) = \frac{1}{2} \left(|z_1|^2, \dots, |z_d|^2\right), \tag{5.20}$$

so that G acts as a subgroup of $\mathbb{T}^d$. Because the action of $\mathbb{T}^d$ commutes with the action of G, there is a residual Hamiltonian action of $\mathbb{T}^d$ on M_α. This action is not effective: the subgroup G acts trivially on M_α. However, the quotient group

$$K = \mathbb{T}^d/G$$

acts faithfully on M_α.

Since the $\mathbb{T}^d$-moment map (5.20) is G-invariant, it descends to a map $\overline{J}\colon M_\alpha \to \mathbb{R}^d$. This map is a moment map for the (non-effective) action of $\mathbb{T}^d$ on M_α: Hamilton's equation for $\overline{J}$ follows immediately from Hamilton's equation for J. The image of this map is the polytope

$$\Delta_\alpha = \{s \in \mathbb{R}^d \mid s_j \geq 0 \text{ for all } j, \text{ and } \sum s_j\alpha_j = \alpha\}, \tag{5.21}$$

which is the intersection of the orthant

$$\mathbb{R}^d_+ = \{s \in \mathbb{R}^d \mid s_j \geq 0 \text{ for } j = 1, \ldots, d\}$$

and the affine space

$$A(\alpha) = \{x \in \mathbb{R}^d \mid \sum x_j\alpha_j = \alpha\}.$$

By applying an appropriate translation, we can ensure that the moment map $\overline{J}$ takes values in the target space $\mathfrak{k}^*$. Since K is a quotient group of $\mathbb{T}^d$, the target space $\mathfrak{k}^*$ is a subspace of $(\mathbb{R}^d)^* = \mathbb{R}^d$. In fact, this subspace is just

$$A(0) = \left\{x \in \mathbb{R}^d \mid \sum x_j\alpha_j = 0\right\}. \tag{5.22}$$

To identify the polytope (5.21) with a polytope that actually lies in the space (5.22), one has to specify the value of the K-moment map at some point, $p \in M_\alpha$, and subtract from it the value of the $\mathbb{T}^d$-moment map at p. Let this difference be ν. Then, as a subset of $\mathfrak{k}^*$, i.e., of the space (5.22), the image of the K-moment map is

$$\Delta = \Delta_\alpha - \nu.$$

Let us describe this image as a polytope in $\mathbb{R}^n$. The quotient $K = \mathbb{T}^d/G$ is a compact connected abelian Lie group. Hence it is isomorphic to $\mathbb{T}^n$ for $n = d - \dim G$. Let us fix an isomorphism between K and $\mathbb{T}^n$. Then we have an exact sequence

$$1 \longrightarrow G \xrightarrow{\rho} \mathbb{T}^d \xrightarrow{\pi} \mathbb{T}^n \longrightarrow 1,$$

where $\rho\colon G \to \mathbb{T}^d$ is the inclusion map and $\pi\colon \mathbb{T}^d \to \mathbb{T}^d/K = \mathbb{T}^n$ is the quotient map. On the infinitesimal level, this becomes

$$0 \longrightarrow \mathfrak{g} \xrightarrow{(\alpha_1,\ldots,\alpha_d)} \mathbb{R}^d \xrightarrow{\pi_*} \mathbb{R}^n \longrightarrow 0,$$

where $-\alpha_j$ are the weights for the G-action on $\mathbb{C}^d$, and, dually,

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\pi_*} \mathbb{R}^d \xrightarrow{L} \mathfrak{g}^* \longrightarrow 0,$$

where

$$L(x_1, \ldots, x_d) = \sum x_i\alpha_i.$$

Let us translate the moment polytope $\Delta_\alpha \subset \mathbb{R}^d_+$ into $\mathbb{R}^n$, as described above: pick any value $\nu \in \Delta_\alpha$; then the moment polytope in $\mathbb{R}^n$ is

$$\Delta = \{y \in \mathbb{R}^n \mid \pi^*(y) + \nu \in \mathbb{R}^d_+\},$$

which is equal to

$$\{y \in \mathbb{R}^n \mid \langle e_i, \pi^*(y) + \nu\rangle \geq 0 \text{ for all } i = 1, \ldots, d\},$$

where e_i are the standard basis elements. Finally, this polytope is equal to the intersection of half-planes:

$$\Delta_{u,\lambda} = \bigcap_{i=1}^{d}\{y \in \mathbb{R}^n \mid \langle u_i, y\rangle + \lambda_i \geq 0\}, \tag{5.23}$$

where $\lambda_i = \langle e_i, \nu \rangle$ and

$$u_i = \pi_*(e_i).$$

5.2. The symplectic Delzant construction. Every convex polytope Δ in $\mathbb{R}^n$ can be written in the form (5.23), where u_i are the normal vectors to the facets. Hence, every convex polytope can be expressed as an "affine slice of $\mathbb{R}^d_+$", by reversing the above construction.

Moreover, if the polytope is integral, that is, if the u_i can be chosen in $\mathbb{Z}^n$, the polytope actually arises as the moment polytope for a reduction of $\mathbb{C}^d$. Namely, define G to be the kernel of the projection map

$$\pi\colon \mathbb{T}^d \to \mathbb{T}^n$$

whose linearization

$$\pi_*\colon \mathbb{R}^d \to \mathbb{R}^n$$

sends the basis element e_i to the vector u_i. Let

$$\Phi\colon \mathbb{C}^d \to \mathfrak{g}^*$$

be the corresponding moment map. Let $\alpha \in \mathfrak{g}^*$ be the image of $\lambda \in \mathbb{R}^d$ under the natural projection $L\colon \mathbb{R}^d \to \mathfrak{g}^*$ that is dual to the inclusion map $G \hookrightarrow \mathbb{T}^d$. Then the quotient $\mathbb{T}^d/G \cong \mathbb{T}^n$ acts on the reduced space $M_\alpha = \Phi^{-1}(\alpha)/G$ with a moment image $\Delta_\alpha \cong \Delta_{u,\lambda}$.

We will now describe the conditions on the polytope Δ which guarantee that the reduced space M_α is actually a manifold or an orbifold.

Denote by $\mathcal{F}_\Delta$ the collection of subsets I of $\{1, \dots, d\}$ which correspond to faces of Δ. In other words, $I \in \mathcal{F}_\Delta$ if and only if there exists $y \in \Delta$ such that $\langle u_i, y\rangle + \lambda_i = 0$ exactly if $i \in I$.

The polytope (5.23) is called *simple* if for every $I \in \mathcal{F}_\Delta$, the normal vectors u_i, for $i \in I$, are linearly independent. If, in addition, these normal vectors can be extended to a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^n$, the polytope (5.23) is called a *regular* polytope, or, equivalently, a *Delzant* polytope.

REMARK 5.14. To check whether or not a polytope is simple, or is Delzant, it is enough to check the above condition for I's which correspond to vertices of Δ.

PROPOSITION 5.15. *If the integral polytope Δ is simple, the corresponding reduced space M_α is an orbifold. If the polytope is Delzant, the reduced space is a manifold.*

PROOF. In the construction described above, the polytope $\Delta = \Delta_{u,\lambda}$ is translated by the map

$$y \mapsto \pi^*(y) + \lambda$$

to

$$\Delta_\alpha = \{s \in \mathbb{R}^d_+ \mid \sum s_j \alpha_j = \alpha\}.$$

For $y \in \mathbb{R}^n$, we have $\langle u_i, y\rangle + \lambda_i = 0$ if and only if $\pi^*(y) + \lambda$ lies on the coordinate hyperplane $\{s_i = 0\}$ in $\mathbb{R}^d_+$. The moment map pre-image of such a point y consists of elements $(z_1, \dots, z_d)$ of $\Phi^{-1}(\alpha)$ such that $z_i = 0$ exactly if $i \in I$. The $\mathbb{T}^d$-stabilizer of such an element is

$$\mathbb{T}^I = \{\lambda \in \mathbb{T}^d \mid \lambda_i = 1 \ \forall i \notin I\}.$$

Hence, the $\mathbb{T}^d$-stabilizers of points $z \in \Phi^{-1}(\alpha)$ are exactly the subtori $\mathbb{T}^I$ for $I \in \mathcal{F}_\Delta$.

The G-stabilizers which occur in the level set $\Phi^{-1}(\alpha)$ are the intersections $G \cap \mathbb{T}^I$ for $I \in \mathcal{F}_\Delta$. The intersection $G \cap \mathbb{T}^I$ is the kernel of the map $\mathbb{T}^I \to \mathbb{T}^n$ whose linearization sends e_i to u_i for all $i \in I$. If u_i, for $i \in I$, are linearly independent, then $G \cap \mathbb{T}^I$ is finite. Therefore, if the polytope is simple, G acts on $\Phi^{-1}(\alpha)$ with finite stabilizers, and the quotient $\Phi^{-1}(\alpha)/G$ is an orbifold. If the vectors u_i, for $i \in I$, extend to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$, then $G \cap \mathbb{T}^I$ is trivial. Therefore, if the polytope is Delzant, G acts on $\Phi^{-1}(\alpha)$ freely, and the quotient $\Phi^{-1}(\alpha)/G$ is a manifold. □

We have shown that the reduction M_α of $\mathbb{C}^d$ by a closed subgroup of $\mathbb{T}^d$ at a regular value α is a Hamiltonian K-orbifold for $K = \mathbb{T}^d/G \cong \mathbb{T}^n$. We have

$$\dim \mathbb{T}^n = d - \dim G = \frac{1}{2} \dim M_\alpha,$$

that is, M_α is a *Delzant space*. (See Section 2 of Chapter 2.) Suppose that G acts *freely* on the level set $\Phi^{-1}(\alpha)$, so that M_α is a manifold, not an orbifold. Then we have shown that the moment map image of M_α is a Delzant polytope. Moreover, starting from *any* Delzant polytope Δ in $\mathbb{R}^n$, we showed how to produce, by reduction of $\mathbb{C}^d$, a Delzant manifold whose moment polytope is Δ.

By Delzant's classification theorem, a Delzant manifold is determined by its moment map image, which is always a Delzant polytope. This implies that, up to isomorphism, every Delzant manifold can be obtained by the procedure that we described above.

Remark 5.16. The Delzant classification theorem has been generalized to orbifolds by Lerman and Tolman. (See Section 2 of Chapter 2.) It follows that every Delzant *orbifold* can also be obtained from $\mathbb{C}^d$ by reduction.

5.3. The complex Delzant construction. In this section we describe the *complex Delzant construction*, i.e., the construction of a *complex* toric variety starting from a Delzant polytope Δ; see [**Au, Cox1, Cox2**] for further details and applications.

Given Δ, one may construct a subgroup G of $\mathbb{T}^d$ as described above, and take the reduced space $M_\alpha = Z/G$ with $Z = \Phi^{-1}(\alpha)$ where $\Phi\colon \mathbb{C}^d \to \mathfrak{g}^*$ is the G-moment map. Let $G_\mathbb{C} = G \cdot \exp(i\mathfrak{g})$ be the complexification of G. Recall from Section 4 that the "semi-stable" set

$$W = G_\mathbb{C} \cdot Z$$

is an open subset of $\mathbb{C}^d$, and that we have a natural diffeomorphism

$$M_\alpha = Z/G \xrightarrow{\cong} W/G_\mathbb{C} \tag{5.24}$$

which shows that M_α as a complex orbifold. The resulting complex toric variety is the quotient $W/G_\mathbb{C}$. We will now give an explicit description of the open set W of $\mathbb{C}^d$ in terms of the combinatorics of the polytope Δ.

First, let us record which sets of coordinates may vanish simultaneously at the points of Z:

$$\mathcal{F}_\alpha = \{I(z) \mid z \in Z = \Phi^{-1}(\alpha)\}, \qquad \text{where } I(z) = \{i \mid z_i = 0\}. \tag{5.25}$$

We note that this collection is the same as the set of faces of the moment polytope (5.21):

$$\mathcal{F}_\alpha = \mathcal{F}_\Delta.$$

Recall that an *alcove* is a connected component of the set of regular values of Φ contained in the image $\Phi(\mathbb{C}^d)$.

LEMMA 5.17. *The collection $\mathcal{F}_\alpha$ is independent of α in an alcove.*

PROOF. Over an alcove, the moment map Φ is an $(S^1)^d$-invariant proper submersion, and hence an $(S^1)^d$-invariant fibration. In particular, the $(S^1)^d$ orbit types that occur in $\Phi^{-1}(\alpha)$ are the same for all α in the alcove. The $(S^1)^d$ orbit type of z is determined by the set $\{i \mid z_i = 0\}$. □

The complex torus $(\mathbb{C}^\times)^d$ acts on $\mathbb{C}^d$ by coordinate-wise multiplication. The orbits for this action are the sets

$$O_I = \{z \in \mathbb{C}^d \mid z_i = 0 \text{ if and only if } i \in I\} = \{0\}^I \times (\mathbb{C}^\times)^{d \smallsetminus I}$$

for all subsets I of $\{1, \ldots, d\}$. We also consider the open sets

$$W_I = \mathbb{C}^I \times (\mathbb{C}^\times)^{d \smallsetminus I},$$

where $\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$.

The following characterization of the set $W = G_\mathbb{C} \cdot Z$ is crucial:

THEOREM 5.18.

$$W = \bigcup_{I \in \mathcal{F}_\alpha} W_I = \bigcup_{I \in \mathcal{F}_\alpha} O_I. \tag{5.26}$$

Here will reproduce the proof from [**Guil1**, Appendix 1].

PROOF OF THE EASY PART OF THEOREM 5.18. By definition,

$$(\mathbb{C}^\times)^d \cdot Z = \bigcup_{I \in \mathcal{F}_\alpha} O_I.$$

Because $G_\mathbb{C} \subseteq (\mathbb{C}^\times)^d$, this implies that

$$W \subseteq \bigcup_{I \in \mathcal{F}_\alpha} O_I.$$

Also, since $O_I \subseteq W_I$ for each I, we have

$$\bigcup_{I \in \mathcal{F}_\alpha} O_I \subseteq \bigcup_{I \in \mathcal{F}_\alpha} W_I.$$

The remaining inclusions $W_I \subseteq W$ will be proved in Section 5.6. □

The above characterization of the set W of semi-stable points completes the complex Delzant construction. Namely, given a polytope Δ in $\mathbb{R}^n$ with d facets, the combinatorics of its facets determines an open subset W of $\mathbb{C}^d$ by (5.26), the slopes of its facets determine the subgroup G of $\mathbb{T}^d$ as described above, and the corresponding toric variety is the quotient $W/G_\mathbb{C}$.

5.4. Orbicharts on a toric variety. An important consequence of Theorem 5.18 is that it provides an explicit open covering of the toric variety $W/G_{\mathbb{C}}$ which is convenient to work with. Namely, M_α is the union of the open sets $W_I/G_{\mathbb{C}}$, for $I \in \mathcal{F}_\alpha$. Moreover, it is enough to consider only $W_I/G_{\mathbb{C}}$ such that $|I| = n$.

We already know that M_α is an orbifold. We will now exhibit each element of the above covering as an *orbichart* on M_α, that is, the quotient of $\mathbb{C}^n$ by a finite group. Moreover, the intersections of these sets will be finite quotients of products of copies of $\mathbb{C}$ and $\mathbb{C}^\times$. We will deduce, in Chapter 8, that this is a "good covering" for the purpose of computing the Dolbeault cohomology of a toric variety.

Suppose that $I \in \mathcal{F}_\alpha$ and $|I| = n$. Consider the map

$$\psi\colon \mathbb{C}^I \to W_I/G_{\mathbb{C}},$$

obtained as the composition

$$\mathbb{C}^I \to \left(\mathbb{C}^I \times \{1\}^{d \smallsetminus I}\right) \hookrightarrow W_I \to W_I/G_{\mathbb{C}}.$$

This map descends to a smooth one-to-one immersion

$$\mathbb{C}^I/\Gamma_I \to W_I/G_{\mathbb{C}}, \tag{5.27}$$

where

$$\Gamma_I = \{g \in G_{\mathbb{C}} \mid g_i = 1 \text{ for all } i \in d \smallsetminus I\}.$$

We argue that this group is finite. Indeed, because $I \in \mathcal{F}_\alpha$ and by the definition of $\mathcal{F}_\alpha$, there exists $z \in Z = \Phi^{-1}(\alpha)$ such that $z_i = 0$ exactly if $i \in I$. The stabilizer of z in $G_{\mathbb{C}}$ is precisely Γ_I. Since α is regular, this stabilizer is finite.

Next, consider the group homomorphism

$$G_{\mathbb{C}} \to (\mathbb{C}^\times)^{d \smallsetminus I} \tag{5.28}$$

obtained as the composition of the inclusion $G_{\mathbb{C}} \hookrightarrow (\mathbb{C}^\times)^d$ with the projection $(\mathbb{C}^\times)^d \to (\mathbb{C}^\times)^{d \smallsetminus I}$. Its kernel is precisely Γ_I, and, hence, is finite. By dimension count, the homomorphism (5.28) is also onto. This implies that the map (5.27) is onto. Consequently, this map is a diffeomorphism, and hence an orbichart on M_α, as claimed.

For $|J| < n$, there exists $I \in \mathcal{F}_\alpha$ such that $|I| = n$ and $J \subset I$. This follows, e.g., from the fact that the polytope Δ_α is simple. The diffeomorphism (5.27) identifies the set $W_J/G_{\mathbb{C}}$ with the subset $\left(\mathbb{C}^J \times (\mathbb{C}^\times)^{I \smallsetminus J}\right)/\Gamma_I$ of $\mathbb{C}^I/\Gamma_I$.

5.5. Regular moment values. Let us digress to examine the moment map (5.19) more closely. Recall that a point z is regular for Φ exactly if its stabilizer is discrete. When checking this condition we may restrict our attention to the identity component of G. The action of an element $\exp\xi$ of G, for $\xi \in \mathfrak{g}$, is given by

$$(\exp\xi)\colon (z_1, \ldots, z_d) \mapsto \left(e^{-i\alpha_1(\xi)} z_1, \ldots, e^{-i\alpha_d(\xi)} z_d\right).$$

The stabilizer of z is

$$\{\exp\xi \mid e^{i\alpha_j(\xi)} = 1 \text{ whenever } z_j \neq 0\}. \tag{5.29}$$

Let I be the subset of $\{1, \ldots, d\}$ such that $z_j = 0$ if and only if $j \in I$. The stabilizer (5.29) is discrete if and only if the weights α_j, for $j \in I$, span $\mathfrak{g}^*$. This implies the following characterization of the set of regular values of Φ. Set $s_j = |z_j|^2$.

LEMMA 5.19. *An element $\alpha \in \mathfrak{g}^*$ is a regular value of Φ if and only if for every subset J of $\{1, \dots, d\}$, if the equation*

$$\alpha = \sum_{j \in J} s_j \alpha_j, \qquad s_j > 0 \tag{5.30}$$

has a solution, then the weights α_j, $j \in J$, span $\mathfrak{g}^*$.

Recall that α is regular if and only if G acts locally freely on the level set $Z = \Phi^{-1}(\alpha)$, and that, in this case, the reduced space $M_\alpha = Z/G$ is an orbifold. One is sometimes interested in the "super-regular" values of Φ, defined as the values α such that G acts *freely* on $Z = \Phi^{-1}(\alpha)$, so that $M_\alpha = Z/G$ is a *manifold*. A similar argument gives the following characterization of super-regular values, when G is a torus:

LEMMA 5.20. *An element $\alpha \in \mathfrak{g}^*$ is a super-regular element of Φ if and only if for every subset J of $\{1, \dots, d\}$, if the equation* (5.30) *has a solution, then the weights α_j, $j \in J$, generate the weight lattice $\mathbb{Z}_G^*$.*

We warn the reader that, whereas the set of regular values is open and dense in $\mathfrak{g}^*$ (by Sard's theorem combined with the properness of Φ), this is not true for the set of super-regular values. This set is open, but it might even be empty. In fact, the mere existence of a super-regular value imposes rather severe restrictions on the weights $\alpha_1, \dots, \alpha_d$.

5.6. The semi-stable points in $\mathbb{C}^\mathbf{d}$. In this section we complete the proof of Theorem 5.18. We first show the second equality of (5.26):

LEMMA 5.21.

$$\bigcup_{I \in \mathcal{F}_\alpha} W_I = \bigcup_{I \in \mathcal{F}_\alpha} O_I.$$

PROOF. Because

$$W_I = \bigcup_{J \subseteq I} O_J,$$

it suffices to prove that if $I \in \mathcal{F}_\alpha$ and $J \subseteq I$, then also $J \in \mathcal{F}_\alpha$. Indeed, suppose $I \in \mathcal{F}_\alpha$. Let $z \in \Phi^{-1}(\alpha)$ be such that $z_i = 0$ exactly if $i \in I$. By Lemma 5.17 and because the alcove is open, there exists a neighborhood U of z such that for all $\alpha' \in \Phi(U)$ we have $\mathcal{F}_{\alpha'} = \mathcal{F}_\alpha$. By perturbing the ith coordinates of z for $i \in I \smallsetminus J$, we obtain an element z' in U such that $z'_i = 0$ exactly if $i \in J$. Then for $\alpha' = \Phi(z')$ we have $J \in \mathcal{F}_{\alpha'} = \mathcal{F}_\alpha$. $\square$

The rest of this section is devoted to completing the proof of Theorem 5.18.

PROOF OF THEOREM 5.18. It remains to prove that $W_I \subseteq W$ for all $I \in \mathcal{F}_\alpha$. By Lemma 5.21, it is enough to prove that $O_I \subseteq W$ for all $I \in \mathcal{F}_\alpha$. The argument below follows closely [**Guil1**, Appendix 1].

Pick any $I \in \mathcal{F}_\alpha$ and any $z \in O_I$, so that $z_j = 0$ exactly if $j \in I$. Because $W = G_\mathbb{C} \cdot \Phi^{-1}(\alpha)$, we need to show that there exists $a \in G_\mathbb{C}$ such that $\Phi(a \cdot z) = \alpha$. The action of the element $\exp(i\eta)$ of $G_\mathbb{C}$, for $\eta \in \mathfrak{g}$, is given by

$$(\exp(i\eta) \cdot z)_j = e^{-i\langle \alpha_j, i\eta \rangle} z_j = e^{\langle \alpha_j, \eta \rangle} z_j.$$

Writing $a = b\exp(i\eta)$ with $b \in G$ and $\eta \in \mathfrak{g}$, we get

(5.31)
$$\Phi(a\cdot z) = \frac{1}{2}\sum_{j=1}^{d} |(a\cdot z)_j|^2\,\alpha_j = \frac{1}{2}\sum_{j=1}^{d} |(\exp(i\eta)\cdot z)_j|^2\,\alpha_j = \frac{1}{2}\sum_{j\in J} e^{2\langle\alpha_j,\eta\rangle} s_j\alpha_j,$$

where $s_j = \frac{1}{2}|z_j|^2 > 0$ for each $j \in J := \{1,\ldots,d\}\smallsetminus I$. By the definition of $\mathcal{F}_\alpha$, there exists $z' \in Z$ such that $z'_j = 0$ exactly if $j \in I$. Hence,

$$\alpha = \Phi(z') = \frac{1}{2}\sum_{j=1}^{d} |z'_j|^2\alpha_j = \sum_{j\in J} s'_j\alpha_j, \tag{5.32}$$

where $s'_j = \frac{1}{2}|z'_j|^2 > 0$ for each $j \in J$. Thus, α is in the open polyhedral cone

$$\Big\{ \sum s'_j\alpha_j \mid s'_j > 0\ ,\ j \in J \Big\}. \tag{5.33}$$

Hence, it is enough to show that the image of the map

$$f(\zeta) = \sum_{j\in J} e^{\langle\alpha_j,\zeta\rangle} s_j\alpha_j \tag{5.34}$$

contains the cone (5.33). (The image of (5.34) is clearly contained in the cone (5.33); we need the opposite inclusion.)

Our first observation is that $f(\zeta)$ is the Legendre transformation of the function

$$F\colon \mathfrak{g} \to \mathbb{R}, \qquad F(\zeta) = \sum_{j\in J} e^{\langle\alpha_j,\zeta\rangle} s_j.$$

In other words,

$$f(\zeta) = dF|_\zeta, \tag{5.35}$$

when we identify $\mathfrak{g}^* = T^*_\zeta\mathfrak{g}$.

Next, we note that the function F is strictly convex, i.e., that the Hessian d^2F is positive definite everywhere. Indeed,

$$d^2|_\zeta F = \sum_{j\in J} e^{\langle\alpha_j,\zeta\rangle} s_j\alpha_j\otimes\alpha_j. \tag{5.36}$$

Because the coefficients $e^{\langle\alpha_j,\zeta\rangle}s_j$ are positive, to show that the bilinear form (5.36) is positive definite, it suffices to show that the weights α_j, $j \in J$, generate $\mathfrak{g}$. This follows immediately from Lemma 5.19.

LEMMA 5.22. *Suppose that F is strictly convex. Then F has a critical point if and only if $F(\zeta) \underset{\zeta\to\infty}{\longrightarrow} \infty$.*

PROOF. See [**Guil1**, Appendix 1, Theorem 3.2]. □

The value α is in the image of (5.34) if and only if there exists ζ such that $dF|_\zeta = \alpha$, by equation (5.35). Set $F_\alpha(\zeta) = F(\zeta) - \alpha(\zeta)$. Then

$$dF_\alpha = dF - \alpha \quad \textit{and} \quad d^2F_\alpha = d^2F.$$

From the first equality we see that α is in the image of (5.34) if and only if F_α has a critical point. From the second equality we conclude that F_α is strictly convex, because F is. By (5.22), F_α has a critical point if and only if

$$F_\alpha(\zeta) \underset{\zeta\to\infty}{\longrightarrow} \infty. \tag{5.37}$$

Substituting (5.32) into the expression for F_α, we have

$$F_\alpha(\zeta) = \sum_{j\in I} e^{\langle \alpha_j, \zeta\rangle} a_j - b_j \alpha_j(\zeta).$$

The jth summand approaches ∞ as ζ escapes $\ker \alpha_j$, by the following fact.

The function

$$g\colon \mathbb{R} \to \mathbb{R}, \qquad g(x) = ae^x - bx$$

satisfies $g(x) \underset{x\to\pm\infty}{\longrightarrow} \infty$ if a and b are positive numbers.

Because $\bigcap_{j\in J} \ker \alpha_j = \{0\}$ (by Lemma 5.19), the desired asymptotic behavior (5.37) holds. □

6. Cobordism of reduced spaces

As we have already pointed out, the fact that "cobordism commutes with reduction" can be used to establish a number of results on the global geometry of Hamiltonian torus actions on symplectic manifolds (cf. [**GGK1**]). A similar fact holds for *abstract* moment maps:

THEOREM 5.23. *Let (M,Ψ) and (M',Ψ') be cobordant manifolds with G-actions and proper abstract moment maps. Let $\alpha \in \mathfrak{g}^*$ be a value that is regular for both Ψ and Ψ'. Then the corresponding reductions, M_α and M'_α, are cobordant (through orbifolds).*

PROOF. Let $(W,\tilde\Psi)$ be a cobording manifold with proper moment map.

If α is also a regular value for $\tilde\Psi$, the quotient $\tilde\Psi^{-1}(\alpha)/G$ gives a cobordism between M_α and M'_α.

If α is not a regular value, we choose a β which is a regular value for $\tilde\Psi$ and which is close enough to α so that the entire interval $[\alpha,\beta]$ consists of values that are regular for both Ψ and Ψ'. This is possible because the set of regular values of $\tilde\Psi$ is dense (by Sard's theorem) and the sets of regular values of Ψ and Ψ' are open (because these maps are proper).

Since β is regular for $\tilde\Psi$, the orbifold $\tilde\Psi^{-1}(\beta)/G$ is a cobordism between M_β and M'_β.

Because the interval $[\alpha,\beta]$ consists of regular values, the level sets $\Psi^{-1}(\alpha)$ and $\Psi^{-1}(\beta)$ are equivariantly diffeomorphic. This follows from Ehresmann's lemma (that a proper submersion is a fibration), which remains valid in the presence of a group action. Dividing by the G-action, we see that the reduced spaces M_α and M_β are diffeomorphic. Similarly, M'_α and M'_β are diffeomorphic.

As a consequence,

$$M_\alpha \cong M_\beta \sim M'_\beta \cong M'_\alpha.$$

Because diffeomorphic spaces are cobordant, and because the cobordism relation is transitive, the reduced spaces M_α and M'_α are cobordant. □

Theorem 5.23 continues to hold in the presence of additional structures. More specifically, we may consider one or more or the following additional structures on (M,Ψ):

(1) An orientation;
(2) An equivariant stable complex structures, J;
(3) A closed two-form, ω, for which Ψ is a genuine moment map;
(4) An equivariant cohomology class, c;

(5) An equivariant complex line bundle, $\mathbb{L}$.

Each of these structures gives rise to a structure of the same type on the reduced space M_α, as follows:

(1) The reduced orientation (see Section 2.2);
(2) The reduced stable complex structure, J_α, (see Section 2.3);
(3) The reduced two-form ω_α (see Section 1);
(4) The reduced cohomology class c_α (see Section 2);
(5) The reduced line bundle $(\mathbb{L}|_Z)/G$, or, more generally, the reduced line bundle $\mathbb{L}_\gamma = (\mathbb{L}|_Z \otimes \mathbb{C}_{-\gamma})/G$ where $\gamma \in \mathbb{Z}^*_G$ is any integral weight.

Suppose that M' carries a structure of the same type. Moreover, suppose that the cobordism between M and M' also carries a structure of the same type, extending the given structures on M and M'. Then the cobordism of Theorem 5.23, between M_α and M'_α, will carry the same type of a structure, giving a cobordism between the structures on M_α and M'_α. This is seen by easy adaptations of the proof of Theorem 5.23. (In the case (3) of a two-form, one should also apply the Duistermaat–Heckman theorem (Theorem 5.6) to express $[\omega_\alpha]$ through $[\omega_\beta]$ and use the fact that the cobordism class of ω_α only depends on its cohomology class.)

Corollary 5.24. Consider triples (M, Ψ, c), where M is a manifold with a G-action, $\Psi\colon M \to \mathfrak{g}^*$ a proper abstract moment map, and $c \in H^*_G(M)$ an equivariant cohomology class. Let $c_\alpha \in H^*(M_\alpha)$ be the reduced cohomology class for a regular value $\alpha \in \mathfrak{g}^*$. Then the integral

$$\int_{M_\alpha} c_\alpha$$

is an invariant of cobordism of the triple (M, Ψ, c).

Proof. This follows from Stokes' theorem applied to the orbifold that gives a cobordism of the reduced spaces. □

Corollary 5.24, applied to the equivariant cohomology classes of $(\omega - \Phi - \alpha)^d/d!$ on M, implies that the Duistermaat–Heckman function is an *invariant* of cobordism. Alternatively, this fact follows from Theorems 2.24 and 5.8.

7. Jeffrey–Kirwan localization

Let $\Phi\colon M \to \mathfrak{g}^*$ be a moment map for a G-action on a compact symplectic manifold. Suppose that $M_{\text{red}} = \Phi^{-1}(0)/G$ is a regular reduced space. An equivariant cohomology class $[c] \in H^*_G(M)$ descends to an ordinary cohomology class $[c_{\text{red}}] \in H^*(M_{\text{red}})$; see Section 2. An important property of the Kirwan map

$$\kappa\colon H^*_G(M) \to H^*(M_{\text{red}}), \qquad [c] \mapsto [c_{\text{red}}]$$

is that it is *onto*. Therefore,

$$H^*(M_{\text{red}}) = M^*_G(M)/\ker \kappa.$$

Often, we want to compute the cohomology ring of a space which arises by symplectic reduction, M_{red}, in terms of the cohomology of M which is often more tractable. For instance, the moduli space of flat connections of a principal bundle arises in this way (cf., for instance, [**Jef, Hue**].) In fact, the study of this moduli space has been one of the motivations for the investigation of cohomological properties of Hamiltonian group actions. Here we concentrate on situations where M_{red} is a manifold, or, at worse, an orbifold. By the Poincaré duality, the kernel of the Kirwan map is determined by the linear functional

$$I\colon [c] \mapsto \int_{M_{\rm red}} [c_{\rm red}]. \tag{5.38}$$

Indeed, $[c]$ is in the kernel if and only if $I([c] \cup [c']) = 0$ for every $[c']$. With this motivation, one would like to obtain a formula for the Kirwan numbers

$$\int_{M_{\rm red}} [c_{\rm red}] \quad , \quad [c] \in H^*_G(M). \tag{5.39}$$

Jeffrey and Kirwan [**JK1**] derived such a formula explicitly. They worked with Hamiltonian actions of compact non-abelian groups on compact symplectic manifolds, and expressed the integral (5.39) by an explicit formula that only involves the data at the set of points that are fixed under the action of a maximal torus.

Then Guillemin gave a topological interpretation of the Jeffrey–Kirwan formula in the case that G is a torus. Shaun Martin showed, by topological means, that the Jeffrey–Kirwan localization formula for non-abelian groups would follow from such a formula for abelian groups. See [**Mart2**]. We will now present an approach to the Jeffrey–Kirwan localization based on our cobordism techniques; cf. [**GGK1**].

Suppose that M is compact, or, more generally, that the moment map Φ is η-polarized for some $\eta \in \mathfrak{g}$, i.e., $\langle \Phi, \eta \rangle$ is proper and bounded from below. By the Linearization Theorem (Theorem 4.11),

$$(M, \omega, \Phi, [c]) \sim \bigsqcup_{F \in \pi_0(M^\eta)} (NF, \omega_F^{\#}, \Phi_F^{\#}, [c_F]).$$

The Kirwan number (5.39) is an invariant of cobordism (Corollary 5.24). This follows from the "cobordism commutes with reduction" theorem (Lemma 5.23) combined with Stokes' theorem. Hence,

$$\int_{M_{\rm red}} c_{\rm red} = \sum_{F \in \pi_0(M^\eta)} \int_{(NF)_{\rm red}} (c_F)_{\rm red}.$$

Notice that the F-summand is nonzero only if $\langle \Phi(F), \eta \rangle < \langle a, \eta \rangle$.

For this formula to be useful, we need to be able to compute the integrals on the right-hand side. The simplest case is when the torus G acts with isolated fixed points p, so that $NF = T_pM$ is a vector space. If the polarizing vector η is chosen generic, we have

$$\int_{M_{\rm red}} c_{\rm red} = \sum_{p \in M^G} \int_{(T_pM)_{\rm red}} (c_p)_{\rm red}.$$

The space $(T_pM)_{\rm red}$, being the reduction of a linear space under a linear torus action, is a toric variety, corresponding to the polytope Δ_p. (See Section 5). One can explicitly express the Kirwan numbers (5.39) in terms of the polytopes Δ_p. This method was used in [**Guil2**] to compute the Riemann–Roch number of $M_{\rm red}$. Metzler applied this method in [**Met1, Met2**] to give explicit formulas for topological invariants of $M_{\rm red}$ such as the signature, the Poincaré polynomial, and the Euler characteristic.

In fact, the cobordism method we just outlined applies under more general assumptions. Namely, let $\Psi\colon M \to \mathfrak{g}^*$ be an abstract moment map, associated with an action of a torus G and let $\alpha \in \mathfrak{g}^*$ be a regular value for Ψ. Consider the reduced space $M_{\rm red} = \Psi^{-1}(\alpha)/G$. The Kirwan map $H^*_G(M) \to H^*(M_{\rm red})$ is no longer onto, however, it is well defined, as are the Kirwan numbers (5.39). The above arguments still allow one to compute these numbers. (Note that in this case one should invoke the Linearization theorem 4.12 for abstract moment maps rather than the Hamiltonian Linearization theorem.)

8. Cutting

Lerman's *symplectic cutting* [**Ler1**] is a simple and versatile operation on Hamiltonian G-manifolds. Given a Hamiltonian S^1-manifold and moment map $\Phi\colon M \to \mathbb{R}$ with regular value α, the Lerman construction "cuts" M into two Hamiltonian S^1-manifolds, M_+ and M_-. Topologically, M_+ (resp., M_-) is obtained from the pre-image $\Phi^{-1}(\alpha,\infty)$ (resp., $\Phi^{-1}(-\infty,\alpha)$) by collapsing its boundary, which is the level set

$$Z = \Phi^{-1}(\alpha),$$

along the S^1-orbits. As a result, we have decompositions

$$M_+ = \Phi^{-1}(\alpha,\infty) \sqcup M_\alpha \quad \text{and} \quad M_- = \Phi^{-1}(-\infty,\alpha) \sqcup M_\alpha, \tag{5.40}$$

where $M_\alpha = Z/S^1$ is the reduced space. In fact, in each of the decompositions (5.40), the pieces on the right-hand side fit together smoothly, i.e., M_+ and M_- are smooth orbifolds. Moreover, M_+ and M_- inherit the structure of Hamiltonian S^1-manifolds, and, in the presence of a G-action on M that commutes with the S^1-action, the G-action descends to M_+ and M_-.

The most common variety of symplectic cutting is that for a Hamiltonian G-manifold M, where G a torus and Φ is the moment map for a sub-circle $S^1 \subset G$. We may then repeat the construction using other sub-circles and end up with an orbifold which coincides with M on the preimage of a polyhedral region Δ in $\mathfrak{g}^*$ over the interior of Δ and with appropriate "collapsing" of M over the boundary of Δ.

In most applications, one starts with a compact polytope $\Delta \subset \mathfrak{g}^*$. Then taking larger and larger Δ's, one obtains successive "compact approximations" of M. Our original version of the linearization theorem, given in [**GGK1**], was stated in terms of such compact approximations; this was before non-compact cobordisms were introduced.

If Δ is a proper convex polyhedral cone, then there exists a linear projection $\Delta \to \mathbb{R}$, given by $\beta \mapsto \langle \beta, \eta \rangle$ for some $\eta \in \mathfrak{g}$, which is proper and bounded from below. The resulting cut manifold is then η-polarized. By appropriately dividing $\mathfrak{g}^*$ into a finite number of proper convex polyhedral cones, we can cut any manifold M with proper moment map into finitely many pieces, each of which is η-polarized for some η. (This fact was used in Remark 4.1.)

Let us assume, for simplicity, that the regular value at which we cut is $\alpha = 0$.

To define the symplectic forms on M_+ and M_- one can invoke symplectic reduction. For instance, M_+ is constructed as the reduction of $M \times \mathbb{C}$ with respect to the diagonal S^1-action

$$M_+ = \left\{(m,z) \in M \times \mathbb{C} \mid \Phi(m) - |z|^2 = 0\right\}/S^1.$$

The inclusion maps of $\Phi^{-1}(0,\infty)$ and M_0 into M_+ are given by $m \mapsto \left(m, \sqrt{\Phi(m)}\right)$ and $[m] \mapsto [m,0]$.

Cutting can also be performed with an *abstract* moment maps and with various kinds of additional structures: an orientation, a stable complex structure, a compatible two-form, an equivariant cohomology class, or a line bundle. Recall that in the previous sections we showed that the reduction procedure applies to each of the these structures. Furthermore, each of these structures on M naturally extends to $M \times \mathbb{C}$, and, hence, induces a similar kind of a structure on the cut-spaces M_+ and M_-.

REMARK 5.25. Reduction and cutting can also be defined for Spin^c structures. The reduction of a Spin^c structure is rather straight-forward once we have the "destabilization" procedure of Proposition D.55. Cutting of a Spin^c structure is more involved and gives a slightly different result than might be expected: let $\mathbb{L}$ and $\mathbb{L}_+$ be the determinant line bundles for the Spin^c structures on M and on M_+. Their restrictions to the open set $\Phi^{-1}(0,\infty)$ coincide. However, the restriction of $\mathbb{L}_+$ to M_0 is not equal to $\mathbb{L}_{\text{red}} := \mathbb{L}|_Z/S^1$, but, rather, to $\mathbb{L}_{\text{red}} \otimes N$, where $N := Z \times_{S^1} \mathbb{C}$. See [**CKT**, Section 6].

In the context of this book, it is important to note that we have a cobordism

$$M \sim M_+ \sqcup M_-. \tag{5.41}$$

To see this, we first focus on one orbit in Z. Assume, for simplicity, that this is a free orbit. Its neighborhood in M can be identified with $U \times S^1 \times \mathbb{R}$, where U is a neighborhood in $Z/S^1 = M_0$, where S^1 acts on the second factor only, and where $\mathbb{R}$ is the coordinate in the normal direction to Z. The orbit in Z gives a point in M_0. Its neighborhood in M_+ (resp., M_-) is $U \times \mathbb{C}$ (resp., $U \times \overline{\mathbb{C}}$). On a neighborhood of a free orbit in Z, the cobordism (5.41) takes the form

$$U \times C \sim (U \times \overline{\mathbb{C}}) \sqcup (U \times \mathbb{C}),$$

where $C = S^1 \times \mathbb{R}$. In Example 3.35 of Chapter 3 we showed that

$$C \sim \overline{\mathbb{C}} \sqcup \mathbb{C}. \tag{5.42}$$

The cobordism (5.41) is obtained by performing the cobordism (5.42) fiberwise over M_0.

We now give an explicit construction for the cutting cobordism (5.41). Let

$$\tilde{W} = \left\{(h, m, z) \in \mathbb{R} \times M \times \mathbb{C} \mid -1 \le h^2 - |z|^2 \le 1 \text{ and } \Phi(m) = \rho(h)\right\}, \tag{5.43}$$

where $\rho(h)$ is a pre-chosen even function such that $\rho(h) = 0$ for $-1 \le h \le 1$ and $\rho'(h) > 0$ for all $h > 1$. It is not hard to check that $(1,0)$ and $(-1,0)$ are regular values of the functions $h^2 - |z|^2$ and $\Phi(m) - \rho(h)$. Hence, $\tilde{W}$ is an orbifold with boundary

$$\partial \tilde{W} = \left\{h = \sqrt{|z|^2 + 1}\right\} \bigsqcup \left\{h = -\sqrt{|z|^2 + 1}\right\} \bigsqcup \left\{|z| = \sqrt{h^2 + 1}\right\}.$$

Let S^1 act on W diagonally, on the M and $\mathbb{C}$ factors. It is not hard to check that this action is locally free. On the quotient orbifold

$$W = \tilde{W}/S^1$$

we take the S^1-action induced by the S^1-action on the M-factor of (5.43) and the abstract moment map $[h, m, z] \mapsto \Phi(m)$. It is not hard to see that the boundary components of W are isomorphic to M_+, M_-, and M, respectively.

Finally, we warn the reader that the cutting cobordism does not carry an equivariant cohomology class or a stable complex or Spin^c-structure. In the presence of such structures on M, one has

$$M \sim M_+ \sqcup M_- \sqcup B$$

where B is an S^2-bundle over the reduced space M_0, equipped with a constant abstract moment map.