CHAPTER 7

The quantum version of the linearization theorem

The main goal of this chapter is to present a quantized version of the linearization theorem. The linearization theorem "decomposes" a symplectic manifold, equipped with a Hamiltonian action with isolated fixed points, into a disjoint union of vector spaces:

$$M \sim \bigsqcup_{p \in M^G} T_p M. \tag{7.1}$$

Quantization of M produces a (virtual) representation $\mathcal{Q}(M)$ of the group G. The Atiyah–Bott formula expresses this representation in terms of the fixed point data in M. We will show that the right-hand side of this formula can be interpreted as the quantization of the spaces T_pM which occur in (7.1). We deduce a quantum version of the linearization theorem:

$$\mathcal{Q}(M) = \sum_{p \in M^G} \mathcal{Q}(T_p M). \tag{7.2}$$

Remark 7.1. This "quantum linearization theorem" can also be obtained directly from the linearization cobordism (7.1). For this, one must utilize an appropriate notion of equivariant index which is defined for non-compact manifolds and which is invariant under proper cobordism. See [**Brav4**].

1. The quantization of $\mathbb{C}^d$

Let M be the linear space $\mathbb{C}^d$ and let ω be the symplectic form

$$\omega_r = \sqrt{-1} \sum_{j=1}^{d} \epsilon_j dz_j \wedge d\bar{z}_j \tag{7.3}$$

with

$$\epsilon_1 = \ldots = \epsilon_r = -1 \quad \text{and} \quad \epsilon_{r+1} = \ldots = \epsilon_d = +1.$$

(That is, $\omega = 2 \sum \epsilon_j dx_j \wedge dy_j$, the factor 2 being a mere convenience.)

Since ω_r is exact and $\mathbb{C}^d$ is simply-connected, the pre-quantization of $\mathbb{C}^d$ poses no problems. By Corollary 6.6, there exists a unique (up to isomorphism) pre-quantization triple $(\mathbb{L}, \langle , \rangle, \nabla)$, the line bundle $\mathbb{L}$ is trivial, and the trivializing section s satisfies $\nabla s = \sqrt{-1}\beta s$, where $\omega = -d\beta$.

Let us now describe the quantization of $\mathbb{C}^d$ with the symplectic form ω_r and the polarization given by the standard complex structure. The quantization line bundle $\mathbb{L} = \mathbb{L}_r$ then becomes a holomorphic line bundle over $\mathbb{C}^d$, and the above trivialization is holomorphic if the primitive β is of type $(1,0)$. Furthermore, the holomorphic structure on $\mathbb{L}_r$ is independent of the choice of such a form β. We choose to work with the one-form

$$\beta = \sqrt{-1} \sum \epsilon_j \overline{z}_j dz_j. \tag{7.4}$$

As in the previous section, consider the $\mathcal{L}^2$ Dolbeault cohomology spaces

$$H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d; \mathbb{L}_r),$$

defined with respect to the standard Hermitian metric on $\mathbb{C}^d$ and the standard Lebesgue measure $(\sqrt{-1})^d dz d\overline{z}$. (Note that, up to scalar factors, the Lebesgue measure is equal to both the Liouville measure on $(\mathbb{C}^d, \omega_r)$ and to the measure induced by the Hermitian metric on $\mathbb{C}^d$.)

Remark 7.2. The metric on $\mathbb{C}^d$ is not part of the quantization data. However, any Hermitian positive-definite inner product on the linear space $\mathbb{C}^d$ would result in essentially the same $\mathcal{L}^2$-quantization. Hence, the structure of a linear space on $\mathbb{C}^d$, the complex linear structure on $\mathbb{C}^d$, and the quantization data (in fact, just ω_r) determine the spaces $H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d; \mathbb{L}_r)$.

The *Bargmann space* $\mathcal{L}^2_{\text{hol}}(\mathbb{C}^d; \mu)$ is the space of entire holomorphic functions on $\mathbb{C}^d$ which are $\mathcal{L}^2$ integrable with respect to the Gaussian measure

$$\mu = (\sqrt{-1})^d e^{-|z|^2} dz d\overline{z}.$$

This space is a Hilbert space, in which the monomials $z_1^{m_1} \cdots z_d^{m_d}$, for non-negative integers $m_i \in \mathbb{Z}_+$, form an orthogonal basis. This space is sometimes referred to as the *Fock space*; see, for instance, [**Man2**] or [**Wd**]. Similarly, the "Bargmann space" $\mathcal{L}^2_{\text{hol}}(\overline{\mathbb{C}}^r \times \mathbb{C}^{d-r}; \mu)$ is the space of functions on $\mathbb{C}^d$ which are anti-holomorphic with respect to the first r coordinates and holomorphic with respect to the last $(d-r)$ coordinates and which are $\mathcal{L}^2$ integrable with respect to μ. The monomials $\overline{z}_1^{m_1} \cdots \overline{z}_r^{m_r} z_{r+1}^{m_{r+1}} \cdots z_d^{m_d}$ form an orthogonal basis for this space.

Theorem 7.3. *The space $H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d; \mathbb{L}_r)$ is zero if $k \neq r$. For $k = r$, this space is the tensor product of the polarized Bargmann space $\mathcal{L}^2_{\text{hol}}(\overline{\mathbb{C}}^r \times \mathbb{C}^{d-r}; \mu)$ with the one-dimensional Hilbert space spanned by*

$$s \otimes d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_r, \tag{7.5}$$

where s is the trivializing section of $\mathbb{L}_r$ described above.

Example 7.4. When $r = 0$, we have the standard Kähler symplectic form $\omega_0 = \sqrt{-1} \sum_{j=1}^d dz_j \wedge d\overline{z}_j$. The spaces $H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d; \mathbb{L}_0)$ vanish for $k > 0$, and $H^{0,0}_{\mathcal{L}^2}(\mathbb{C}^d; \mathbb{L}_0)$ is the Bargmann space $\mathcal{L}^2_{\text{hol}}(\mathbb{C}^d; \mu)$ of $\mathcal{L}^2$ integrable entire functions.

Remark 7.5. Note that the line bundles $\mathbb{L}_r$ are all equivalent to each other (and are all trivial) for various values of r as holomorphic line bundles and as Hermitian line bundles, but not as holomorphic Hermitian line bundles (nor as line bundles with connections). We have already proved the holomorphic triviality. The Hermitian triviality follows immediately from the fact that $\mathbb{C}^d$ is contractible. The Hermitian holomorphic non-equivalence is a consequence of the fact that for different values of r the pairs (ω_r, J) give rise to indefinite metrics of different signatures. Also, arguing as in the proof of Theorem 7.3 below, one can show that a trivializing section s of $\mathbb{L}_r$ cannot simultaneously be holomorphic and have constant norm.

REMARK 7.6. The spaces $H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d;\mathbb{L}_r)$ and their kin have been extensively studied. For example, the representations of $\mathrm{U}(r,d-r)$ on $H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d;\mathbb{L}_r)$ are analyzed in [**Man1**] where Theorem 7.3 has also been proved; see also [**BR, Man2**].

PROOF OF THEOREM 7.3. The plan of the proof is as follows. First we reduce the theorem to a calculation of the $\mathcal{L}^2$ Dolbeault cohomology of $\mathbb{C}^d$ with trivial coefficients, but with respect to a different measure. Then we explicitly carry out the calculations for $d=1$ and $r=0,1$. Finally, the general case is reduced to the latter two via an analogue of the Künneth formula.

Step 1: Reduction to the untwisted $\mathcal{L}^2$ Dolbeault cohomology. Recall that $\mathbb{L}_r$ admits a holomorphic trivializing section s such that $\nabla s=\sqrt{-1}\beta s$, with $\beta=\sqrt{-1}\sum\epsilon_j\overline{z}_j dz_j$. Since

$$\begin{aligned} d\langle s,s\rangle &= \langle\nabla s,s\rangle+\langle s,\nabla s\rangle \\ &= \sqrt{-1}(\beta-\overline{\beta})\langle s,s\rangle \\ &= d\left(-\sum\epsilon_j|z_j|^2\right)\langle s,s\rangle, \end{aligned}$$

up to a multiplicative constant (which we can scale to be equal to one),

$$\langle s,s\rangle=e^{-\sum\epsilon_j|z_j|^2}. \tag{7.6}$$

The mapping

$$f d\overline{z}_{i_1}\wedge\ldots\wedge d\overline{z}_{i_q}\mapsto s\otimes f d\overline{z}_{i_1}\wedge\ldots\wedge d\overline{z}_{i_q}$$

gives rise to an isomorphism of the Dolbeault complexes,

$$\Omega^{0,*}(\mathbb{C}^d)\to\Omega^{0,*}(\mathbb{C}^d;\mathbb{L}_r).$$

This map induces an isomorphism of the $\mathcal{L}^2$–Dolbeault complexes, where the target complex is equipped with the Lebesgue measure $(\sqrt{-1})^d dz d\overline{z}$ and, by (7.6), the complex $\Omega^{0,*}(\mathbb{C}^d)$ is equipped with the measure

$$\mu_r=(\sqrt{-1})^d e^{-\sum\epsilon_i|z_i|^2}\,dz d\overline{z}.$$

Emphasizing its dependence on r, we will denote the latter complex by $\Omega^{0,*}_{\mathcal{L}^2}(\mathbb{C}^d;\mu_r)$ and its cohomology by $H^{0,*}_{\mathcal{L}^2}(\mathbb{C}^d;\mu_r)$. Hence,

$$H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d;\mathbb{L}_r)=H^{0,k}_{\mathcal{L}^2}(\mathbb{C}^d;\mu_r), \tag{7.7}$$

and to prove the theorem it suffices to compute $H^{0,*}_{\mathcal{L}^2}(\mathbb{C}^d;\mu_r)$.

Step 2: $d=1$ and $r=0$. In this case, the measure is

$$\mu_0=\sqrt{-1}e^{-|z|^2}dz d\overline{z}=\mu, \tag{7.8}$$

and the zeroth cohomology is the Bargmann space $\mathcal{L}^2_{\mathrm{hol}}(\mathbb{C};\mu)$. For $k>1$, the kth cohomology is zero, because $\Omega^{0,k}(\mathbb{C})=0$. It remains to show that $H^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_0)=0$. The monomial one-forms $z^k\overline{z}^l d\overline{z}$ form an orthogonal basis of the Hilbert space $\Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_0)$. (To get an orthonormal basis, one must divide by $||z^k\overline{z}^l d\overline{z}||=||z^k\overline{z}^l||=\sqrt{\pi(k+l)!}$.) Since

$$z^k\overline{z}^l d\overline{z}=\overline{\partial}\left(\frac{1}{l+1}z^k\overline{z}^{l+1}\right),$$

the image of $\overline{\partial}$ is dense in $\Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_0)$. By the definition of $\mathcal{L}^2$ cohomology (see Section 5.3 of Chapter 6), $H^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_0)=0$.

Step 3: $d = 1$ and $r = 1$. In this case, the measure is

$$\mu_1 = \sqrt{-1}e^{|z|^2}dzd\overline{z},$$

and the zeroth cohomology is $H^{0,0}(\mathbb{C}; \mathbb{L}_1) = \mathcal{L}^2_{\mathrm{hol}}(\mathbb{C}; \mu_1)$. The assertion that this space is zero is similar in spirit to Liouville's theorem on bounded holomorphic functions. Indeed, this space is the space of holomorphic functions f on $\mathbb{C}$ which satisfy

$$\int |f(z)|^2 e^{|z|^2}\, dzd\overline{z} < \infty. \tag{7.9}$$

In essence, this is a growth condition and, as in the case of Liouville's theorem, a holomorphic function for which this integral is finite must be zero. Formally, however, the integral condition (7.9) does not imply directly that f is bounded, and Liouville's theorem does not apply. So let us give an elementary proof of the fact that an entire function f satisfying (7.9) is necessarily zero.

Consider the integral of $|f|^2$ over the circle of radius r:

$$I(r) = \int \left| f\left(re^{\sqrt{-1}\theta}\right)\right|^2 d\theta = 2\pi \sum |a_n|^2 r^{2n},$$

where $f(z) = \sum a_n z^n$ is the Taylor expansion of f. This integral is a non-negative monotone increasing function in r. On the other hand, (7.9) readily implies that $I(r_k) \to 0$ for some sequence $r_k \to \infty$. Therefore, $I(r) = 0$ for all $r > 0$ and hence $f \equiv 0$.

Since the k-th cohomology vanishes for $k > 1$, it remains to show that there is a unitary isomorphism

$$\mathcal{L}^2_{\mathrm{hol}}(\overline{\mathbb{C}}; \mu) \to H^{0,1}_{\mathcal{L}^2}(\mathbb{C}; \mu_1). \tag{7.10}$$

Consider the homomorphism

$$Q\colon \mathcal{L}^2_{\mathrm{hol}}(\overline{\mathbb{C}}; \mu) \to \Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C}; \mu_1)$$

given by $Q\colon \overline{f} \mapsto \overline{f}e^{-|z|^2}d\overline{z}$.

Lemma 7.7. *Q induces a unitary isomorphism* (7.10) *in cohomology.*

Proof. A simple calculation shows that Q is an isometry. Hence, since the domain of Q is complete, so is its image. As a consequence, the image of Q is closed. Therefore, to show that Q induces an epimorphism in cohomology, it suffices to prove that the image of Q projects to a dense subspace in

$$H^{0,1}_{\mathcal{L}^2}(\mathbb{C}; \mu_1) = \frac{\Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C}; \mu_1)}{\text{closure}\left[\Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C}; \mu_1) \bigcap \overline{\partial}\Omega^{0,0}_{\mathcal{L}^2}(\mathbb{C}; \mu_1)\right]}.$$

Recall that the monomials $z^k\overline{z}^l$ form an orthogonal basis in $\mathcal{L}^2(\mathbb{C}; \mu)$. Therefore, $z^k\overline{z}^l e^{-|z|^2}d\overline{z}$ is an orthonormal basis in $\Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C}; \mu_1)$. For any positive integers k and l, if $k \le l$, the form $z^k\overline{z}^l e^{-|z|^2}$ lies in the same cohomology class as a scalar multiple of the form $\overline{z}^{l-k}e^{-|z|^2}$. This follows by induction from the equality

$$z^k\overline{z}^l e^{-|z|^2}d\overline{z} = lz^{k-1}\overline{z}^{l-1}e^{-|z|^2}d\overline{z} - \overline{\partial}(z^{k-1}\overline{z}^l e^{-|z|^2}).$$

Similarly, when $k > l$, the form $z^k\overline{z}^l e^{-|z|^2}d\overline{z}$ lies in the same cohomology class as a scalar multiple of the form $z^{k-l}e^{-|z|^2}d\overline{z} = -\overline{\partial}(z^{k-l-1}e^{-|z|^2})$. This shows that the forms $\overline{z}^m e^{-|z|^2} d\overline{z} = Q(\overline{z}^m)$, for $m \geq 0$, topologically generate $H^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_1)$. Hence the homomorphism (7.10) induced by Q is onto.

To show that this homomorphism is one-to-one, we will prove that the image of Q is orthogonal to $\Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_1) \cap \overline{\partial}\Omega^{0,0}_{\mathcal{L}^2}(\mathbb{C};\mu_1)$. Recall that the vectors $Q(\overline{z}^m)$, $m \geq 0$, topologically span the image of Q. Hence, it is enough to show that each $Q(\overline{z}^m)$ is orthogonal to $\overline{\partial}(fe^{-|z|^2})$, where f is such that

$$f \in \mathcal{L}^2(\mathbb{C};\mu) \tag{7.11}$$

and

$$\overline{\partial}(fe^{-|z|^2}) \in \Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_1). \tag{7.12}$$

A straightforward calculation shows that in $\Omega^{0,1}_{\mathcal{L}^2}(\mathbb{C};\mu_1)$ the inner product of $Q(\overline{z}^m)$ and $\overline{\partial}(fe^{-|z|^2})$ is given by

$$\left\langle \overline{\partial}(fe^{-|z|^2}), Q(\overline{z}^m)\right\rangle = \sqrt{-1}\int \partial_{\overline{z}}(z^m fe^{-|z|^2})\, dz d\overline{z}.$$

In particular, this integral exists (as a Lebesgue integral) because the inner product is defined owing to (7.12). Hence, denoting by $D(r)$ and $S(r)$ the disc and the circle of radius r centered at the origin, we obtain

$$\begin{aligned} \left\langle \overline{\partial}(fe^{-|z|^2}), Q(\overline{z}^m)\right\rangle &= -\sqrt{-1}\lim_{r\to\infty}\int_{D(r)} \overline{\partial}(z^m fe^{-|z|^2} dz) \\ &= -\sqrt{-1}\lim_{r\to\infty}\int_{S(r)} z^m fe^{-|z|^2}\, dz, \end{aligned} \tag{7.13}$$

where the second equality follows from Stokes' formula. We conclude that it suffices to prove that

$$\lim_{r\to\infty}\int_{S(r)} z^m fe^{-|z|^2} dz = 0. \tag{7.14}$$

A simple calculation relying on the Cauchy–Schwarz inequality shows that

$$\left|\int_{S(r)} z^m fe^{-|z|^2} dz\right| \leq 2\pi r^{m+1} e^{-r^2} F(r)^{\frac{1}{2}}, \tag{7.15}$$

where

$$F(r) = \int_0^{2\pi} \left|f(re^{\sqrt{-1}\theta}\right|^2 d\theta.$$

By (7.11),

$$\int |f|^2\mu = 2\int_0^\infty F(r)e^{-r^2} r\, dr < \infty,$$

and, therefore,

$$F(r_n)e^{-r_n^2} r_n \to 0$$

for some sequence $r_n \to \infty$. From this it is easy to see that

$$r_n^{m+1} e^{-r_n^2} F(r_n)^{\frac{1}{2}} \to 0.$$

Therefore, by (7.15),

$$\left|\int_{S(r_n)} z^m fe^{-|z|^2} dz\right| \to 0$$

as $r_n \to \infty$. Since the limit in (7.14) exists, we conclude from (7.13) that the inner product is zero. This completes the proof of the lemma and the third step of the proof of the theorem. □

Step 4: Arbitrary d and r. The general case follows from the previous cases by the Künneth formula. Namely, there exists an obvious homomorphism of complexes

$$\Omega^{0,*}_{\mathcal{L}^2}(\mathbb{C};\mu_1)^{\otimes r} \otimes \Omega^{0,*}_{\mathcal{L}^2}(\mathbb{C};\mu_0)^{\otimes(d-r)} \to \Omega^{0,*}_{\mathcal{L}^2}(\mathbb{C}^d;\mu_r).$$

One can show that this homomorphism induces an isomorphism in $\mathcal{L}^2$-cohomology. The only non-zero cohomology of the tensor product is in degree r and is equal to $\mathcal{L}^2_{\mathrm{hol}}(\overline{\mathbb{C}};\mu)^{\otimes r} \otimes \mathcal{L}^2_{\mathrm{hol}}(\mathbb{C};\mu)^{\otimes(d-r)}$, (the tensor product being the Hilbert tensor product; see [**Ki2**, Chapter 4] or [**Dou**, Exercise 3.21]). It is not hard to see that the latter space is naturally isomorphic to the Bargmann space $\mathcal{L}^2_{\mathrm{hol}}(\overline{\mathbb{C}}^r \times \mathbb{C}^{d-r};\mu)$. □

Remark 7.8. An argument similar to the proof of Theorem 7.3 allows one to prove a version of Serre's duality of the $\mathcal{L}^2$ cohomology of $\mathbb{L}$. For example, one can show that $H^{d,k}_{\mathcal{L}^2}(M;\mathbb{L})^* \cong H^{0,d-k}_{\mathcal{L}^2}(M;\mathbb{L})$ for the measure μ_0 on $M = \mathbb{C}^d$. Note that both spaces are zero unless $k = d$. (For a general version of Serre's duality for $\mathcal{L}^2$ cohomology see, e.g., [**Schm**, Proposition 9.1].)

Definition 7.9. The quantization of $(\mathbb{C}^d, \omega_r)$, with ω_r given by (7.3), is the virtual vector space:

$$\mathcal{Q}(\mathbb{C}^d,\omega_r) = (-1)^r H^{0,r}(\mathbb{C}^d;\mathbb{L}_r). \tag{7.16}$$

The motivation for this definition is that the quantization of $\mathbb{C}^d$ ought to be the alternating sum

$$\mathcal{Q}(\mathbb{C}^d,\omega_r) = \sum (-1)^i H^{0,r}_{\mathcal{L}^2}(\mathbb{C}^d;\mathbb{L}_r). \tag{7.17}$$

In general, however, this sum might not make sense if it contained more than one infinite–dimensional term. (See Remark 6.36.) However, by Theorem 7.3, we know that the only non-zero term in the sum is the r-th term and we take this term as the definition of $\mathcal{Q}(\mathbb{C}^d,\omega_r)$. Thus, by Theorem 7.3 we conclude

Remark 7.10. To be consistent with the definition of quantization of compact manifolds we should indicate how this quantization of $\mathbb{C}^d$ depends on the orientation. To this end, we declare that (7.16) gives the quantization of $\mathbb{C}^d$ equipped with the standard complex orientation. Then, by definition, for $\mathbb{C}^d$ with the opposite orientation the quantization is $-\mathcal{Q}(\mathbb{C}^d,\omega_r)$. This sign convention will become relevant in Section 4 in the proof of the quantum linearization theorem. Until then we will always assume that $\mathbb{C}^d$ carries its standard complex orientation unless specified otherwise.

Theorem 7.11. *The quantization* (7.17) *is the tensor product of the Bargmann space $\mathcal{L}^2_{\mathrm{hol}}(\overline{\mathbb{C}}^r \times \mathbb{C}^{d-r};\mu)$ with the one-dimensional Hilbert space spanned by*

$$s \otimes d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_r,$$

with the formal sign $(-1)^r$.

We now discuss an equivariant version of these results. Let G be a torus, let $\alpha_i \in \mathfrak{g}^*$, $i = 1, \ldots, d$, be weights of G, and let G act on $\mathbb{C}^d$ as

$$\tau(\exp\xi)z = (e^{-\sqrt{-1}\alpha_1(\xi)}z_1, \ldots, e^{-\sqrt{-1}\alpha_d(\xi)}z_d). \tag{7.18}$$

This action preserves ω_r and the complex structure of $\mathbb{C}^d$. Let us show that this action is pre-quantizable by describing explicitly how it lifts to an action of G on sections of $\mathbb{L}_r$. First of all we note

PROPOSITION 7.12. *The moment map associated with the action τ is of the form*

$$\Phi(z) = \Phi(0) + \sum \epsilon_i |z_i|^2 \alpha_i, \tag{7.19}$$

where $\Phi(0)$ is an arbitrary constant.

PROOF. In polar coordinates, the two-form is $2\sum \epsilon_i r_i dr_i \wedge d\theta_i$, and the infinitesimal action of $\mathfrak{g}$ on $\mathbb{C}^d$ is given by the generating vector fields $\xi_{\mathbb{C}^d} = -\sum \alpha_i(\xi)\frac{\partial}{\partial \theta_i}$ for $\xi \in \mathfrak{g}$. It is easy to check that the function $\Phi(0) + \sum \epsilon_i r_i^2 \alpha_i$ satisfies the moment map equation $\iota(\xi_{\mathbb{C}^d})\omega = d\Phi^\xi$. ☐

Corollary 6.18 gives a one-to-one correspondence between infinitesimal lifts of the action to $\mathbb{L}$ and moment maps Φ. By Theorem 6.7, a necessary and sufficient condition for a moment map to give a lift of the G-action is that $\Phi(0) \in \mathbb{Z}_G^*$. The value $\Phi(0)$ is then the weight of the representation of G on the fiber of $\mathbb{L}$ at the origin. Conversely, if this condition is met, a lift of the action is given by letting G act on the trivializing section s with the weight $\Phi(0)$:

$$\tau(\exp \xi)s = e^{\sqrt{-1}\Phi(0)(\xi)} s. \tag{7.20}$$

Moreover, this pre-quantization of the Hamiltonian G-action (7.18) with moment map (7.19) is unique up to isomorphism by Proposition 6.5 and Corollary 6.18.

Assume now that $\Phi(0) \in \mathbb{Z}_G^*$. Because the G-action on $\mathbb{C}^d$ is holomorphic, one obtains a representation of G on the virtual vector space $\mathcal{Q}(\mathbb{C}^d, \omega_r)$. We emphasize that this representation, $\mathcal{Q}(\mathbb{C}^d, \omega_r, \Phi)$, depends on $\Phi(0)$. With (7.20) defining the action of G on the trivializing section, s, it is now easy to describe $\mathcal{Q}(\mathbb{C}^d, \omega_r, \Phi)$:

THEOREM 7.13. *Theorems 7.3 and 7.11 hold as equalities between virtual representations of G.*

We will describe the representation $\mathcal{Q}(\mathbb{C}^d, \omega_r, \Phi)$ in more detail in Section 3. In particular, we will show that when the weights α_i satisfy a certain positivity requirement, the alternating sum (7.17) defining $\mathcal{Q}(M)$ *does* belong to a suitable completion of the representation ring $R(G)$. (See Remark 6.36.)

2. Partition functions

Let G be a torus and let $\alpha_1, \dots, \alpha_d \in \mathbb{Z}_G^*$ be elements of its integral weight lattice. We associate with the collection $\alpha_1, \dots, \alpha_d$ a function

$$N \colon \mathbb{Z}_G^* \to \mathbb{Z} \cup \{\infty\},$$

called the *partition function.* By definition, for a weight, α, the value $N(\alpha)$ is the number of solutions of the equation

$$\sum_{i=1}^{d} k_i \alpha_i = \alpha, \qquad k_i \in \mathbb{Z}, \qquad k_i \geq 0, \qquad i = 1, \dots, d. \tag{7.21}$$

Geometrically, $N(\alpha)$ is the number of lattice points lying in the polytope

$$\Delta_\alpha = \{s \in \mathbb{R}_+^d \mid \sum s_i \alpha_i = \alpha\},$$

where $\mathbb{R}^d_+$ is the positive orthant

$$\mathbb{R}^d_+ = \{s \in \mathbb{R}^d \,,\, s_i \geq 0 \,,\, i = 1, \dots, d\},$$

i.e.,

$$N(\alpha) = \#(\mathbb{Z}^d \cap \Delta_\alpha).$$

As a consequence, $N(\alpha)$ is approximately equal to the volume of the polytope Δ_α, normalized appropriately.

The polytope Δ_α is the intersection of the positive orthant $\mathbb{R}^d_+$ with the affine subspace

$$A(\alpha) = \{x \in \mathbb{R}^d \mid \sum x_i \alpha_i = \alpha\}. \tag{7.22}$$

Equivalently, Δ_α is a level set of the projection map

$$\mathbb{R}^d_+ \to \mathfrak{g}^*, \qquad s \mapsto \sum s_i \alpha_i. \tag{7.23}$$

Recall from Chapter 4 that the weights $\alpha_i \in \mathfrak{g}^*$ are said to be *polarized* if there exists a vector $\xi \in \mathfrak{g}$ such that

$$\alpha_i(\xi) > 0 \quad \text{for} \quad i = 1, \dots, d. \tag{7.24}$$

To proceed we need a few standard facts from convex geometry.

Proposition 7.14. *Let α_i be elements of the integral weight lattice $\mathbb{Z}^*_G \subset \mathfrak{g}^*$. The following conditions are equivalent to each other:*

(1) *The weights α_j are polarized.*
(2) *The partition function $N(\alpha)$, $\alpha \in \mathbb{Z}^*_G$, assumes only finite values.*
(3) *There exists a weight $\alpha \in \mathbb{Z}^*_G$ such that $N(\alpha)$ is non-zero and finite.*

Proof. If the weights α_j are polarized, the polytopes Δ_α are compact, by Proposition 4.14. This implies that $N(\cdot)$ is finite–valued.

Suppose now that the weights α_j are not polarized. By Proposition 4.14, there exist non-negative coefficients $l_1, \dots, l_d$, not all zero, such that $\sum l_j \alpha_j = 0$. Because the weights α_j are in a lattice, we may assume, without loss of generality, that all l_j are integers. For any α, if $k = (k_1, \dots, k_d)$ is a solution of (7.21), then so is $(k_1 + ml_1, \dots, k_d + ml_d)$ for any non-negative integer m. Hence, if $N(\alpha)$ is non-zero, the value $N(\alpha)$ is infinite. ∎

The equivalence of assertions (1) and (2) will be particularly important for us:

Corollary 7.15. The partition function $N(\cdot)$ takes finite values if and only if the weights α_i are polarized.

We will assume from now on that the weights α_j are polarized. We have the following discrete analogue of Proposition 4.17.

Theorem 7.16. *Assume that the weights $\alpha_i \in \mathbb{Z}^*_G$, $i = 1, \dots, d$, are polarized and let $n = \dim \operatorname{span}\{\alpha_i \mid i = 1, \dots, d\}$. Then there exists a positive constant C such that*

$$N(\alpha) \leq C|\alpha|^{d-n}$$

for all $\alpha \in \mathbb{Z}^*_G$.

PROOF. Without loss of generality, we may assume that the weights α_i span $\mathfrak{g}^*$; otherwise, we replace $\mathfrak{g}^*$ by their span. For $\alpha \in \mathbb{Z}^*_G$, let

$$A(\alpha) = \{x \in \mathbb{R}^d \mid \sum x_i\alpha_i = \alpha\}.$$

This is an affine plane of dimension $d - n$.

Consider the shifting map

$$i\colon (x_1, \dots, x_d) \mapsto (x_1 + 1, \dots, x_d + 1)$$

from $A(\alpha)$ to $A(\beta)$, where $\beta = \alpha + \alpha_1 + \ldots + \alpha_d$.

The map i sends $\mathbb{Z}^d \cap \Delta_\alpha$ into

$$\Delta_\beta = \{x \in \mathbb{R}^d_+ \mid \sum x_i\alpha_i = \beta\}.$$

For each $k \in \mathbb{Z}^d \cap \Delta_\alpha$, the ball in $A(\beta)$ of radius $1/2$ centered at $i(k)$ is still contained in Δ_β. (This, in particular, implies that when $\mathbb{Z}^d \cap \Delta_\alpha \neq \emptyset$, the polytope Δ_β has dimension $d-n$, although Δ_α may have a lower dimension, and that Δ_β spans the affine space $A(\beta)$.) Furthermore, it is easy to see that two such balls, for distinct k and k' in $\mathbb{Z}^d \cap \Delta_\alpha$, are disjoint. Comparing volumes, we see that

$$vN(\alpha) \leq \mathrm{vol}(\Delta_\beta), \tag{7.25}$$

where $v > 0$ is the volume of the $(d-n)$-dimensional ball of radius $1/2$ in $A(\beta)$.

Let ξ be a polarizing vector, so that

$$m := \min\{\alpha_1(\xi), \dots, \alpha_d(\xi)\} > 0.$$

By definition, $\beta = \sum x_i\alpha_i$ for all $x \in \Delta_\beta \subset \mathbb{R}^d_+$. As a consequence,

$$\beta(\xi) = \sum x_i\alpha_i(\xi) \geq m \sum x_i,$$

and so

$$||x||^2 = \sum x_i^2 \leq \Big(\sum x_i\Big)^2 \leq \left(\frac{1}{m}\beta(\xi)\right)^2.$$

Hence,

$$\Delta_\beta \subset A(\beta) \cap B(r),$$

where $B(r)$ is the ball in $\mathbb{R}^d$ of radius $r = \frac{1}{m}\beta(\xi)$. Comparing volumes again, we conclude that

$$\mathrm{vol}(\Delta_\beta) \leq \mathrm{vol}(A(\beta) \cap B(r)).$$

Since the volume on the right-hand side is smaller than that of the $(d-n)$-dimensional ball of radius $r = \frac{1}{m}\beta(\xi)$, it follows from (7.25) that there exists a constant, $C' > 0$, which only depends on the weights α_i, such that

$$N(\alpha) \leq C'(\beta(\xi))^{d-n}.$$

Since $\beta(\xi) = \alpha(\xi) + \mathit{const}$, the theorem follows. □

There is a close relationship between the partition function $N(\cdot)$ associated with the weights $\alpha_1, \dots, \alpha_d \in \mathbb{Z}^*_G$ and the G-action on $\mathbb{C}^d$ with weights $-\alpha_j$ given by

$$(z_1, \dots, z_d) \mapsto (e^{-\sqrt{-1}\alpha_1(\xi)}, \dots, e^{-\sqrt{-1}\alpha_d(\xi)}). \tag{7.26}$$

The action (7.26) has a moment map

$$\Phi(z) = \frac{1}{2}\sum |z_j|^2\alpha_j.$$

The image of this map is the convex polyhedral cone

$$C(\alpha_1,\dots,\alpha_d)=\left\{\sum s_j\alpha_j \mid s\in\mathbb{R}^d_+\right\}.$$

The partition function $N(\cdot)$ is supported in this cone.

Also, the partition function $N(\cdot)$ is supported in the sub-lattice of $\mathbb{Z}^*_G$ that is generated by the lattice elements $\alpha_1,\dots,\alpha_d$. We have the following relation between this lattice and the action of G:

LEMMA 7.17. *The action* (7.26) *is effective if and only if the weights* $\alpha_1,\dots,\alpha_d$ *generate* $\mathbb{Z}^*_G$.

PROOF. The element $\exp\xi\in G$ is in the kernel of (7.26) if and only if $\alpha_j(\xi)\in 2\pi\mathbb{Z}$ for all j. This means that ξ is in the dual lattice to the lattice spanned by $\alpha_1,\dots,\alpha_d$ (with the 2π-conventions of Appendix A). On the other hand, $\exp\xi\in G$ is trivial if and only if ξ is in the lattice $\mathbb{Z}_G=\ker\exp$. Since the dual to $\mathbb{Z}^*_G$ is $\mathbb{Z}_G$, the lemma follows. □

We will now assume that the weights α_j are generators of the weight lattice, $\mathbb{Z}^*_G$, over $\mathbb{Z}$. Otherwise, we may replace $\mathbb{Z}^*_G$ by the lattice generated by the α_j.

We recall that an element $\alpha\in\mathfrak{g}^*$ is a regular value of Φ if and only if it satisfies the following condition (see Lemma 5.19):

For every solution of the equation

$$\alpha=\sum_{l=1}^{m}s_{i_l}\alpha_{i_l}, \tag{7.27}$$

with all $s_{i_l}>0$, the weights $\alpha_{i_1},\dots,\alpha_{i_l}$ span $\mathfrak{g}^*$.

Note that if α is not contained in the cone $C(\alpha_1,\dots,\alpha_d)$, then it is regular by default, because equation (7.27) has *no* solutions. If α is in the cone $C(\alpha_1,\dots,\alpha_d)$, it is regular if and only if it cannot be expressed as the convex combination of fewer than $\dim G$ elements α_i. We will denote the set of regular values by $\mathfrak{g}^*_{\text{reg}}$. Then the set

$$C(\alpha_1,\dots,\alpha_d)\cap\mathfrak{g}^*_{\text{reg}} \tag{7.28}$$

is the complement in $C(\alpha_1,\dots,\alpha_d)$ of the lower dimensional cones generated by subsets of the α_j's.

The connected components of the set (7.28) are open convex polyhedral cones, which we will call *moment cones*. Moreover, if $\xi\in\mathfrak{g}$ is a polarizing vector, so that $\alpha_j(\xi)>0$ for all j, then the intersection of each moment cone with the hyper-plane

$$\alpha(\xi)=1$$

is a convex polytope, Δ, and hence the moment cone itself is the cone over this polytope:

$$\{\lambda\alpha \mid \alpha\in\Delta,\lambda\in\mathbb{R}_+\}.$$

EXAMPLE 7.18. Let $\mathfrak{g}^*=\mathbb{R}^3$, and let $\alpha_1,\dots,\alpha_4$ be the vertices of the unit square in the $z=1$ plane. Then $C(\alpha_1,\dots,\alpha_4)$ is the cone over this square. The regular points in this cone are those of its interior points that do not lie on the diagonal planes. Hence, there are four moment cones.

We will next discuss the behavior of the partition function on each of the moment cones.

Definition 7.19. A function $P\colon \mathfrak{g}^* \to \mathbb{C}$ is a *quasi-polynomial* if it is of the form

$$P(\alpha) = \sum_{k=1}^{N} e^{-\frac{\sqrt{-1}\alpha(\xi_k)}{n_k}} P_k(\alpha), \tag{7.29}$$

where P_k are polynomials, n_k are positive integers, and ξ_k are elements of the group lattice $\mathbb{Z}_G$. The *degree* of P is the maximum of the degree of the P_k's.

In what follows we will be interested mainly in the restriction of P to $\mathbb{Z}_G^*$. Formula (7.29) means that $\mathbb{Z}_G^*$ partitions into the union of affine sublattices, and on each of these sublattices, P is a polynomial.

Theorem 7.20. *Let C be a moment cone. Then there exists a unique quasi-polynomial P_C of degree $d-n$, with the property*

$$P_C(\alpha) = N(\alpha) \tag{7.30}$$

for all $\alpha \in \mathbb{Z}_G^* \cap C$.

In Chapter 8 we will interpret the partition function $N(\cdot)$ as the quantization of a Kähler toric variety that is obtained from $\mathbb{C}^d$ by reduction. This variety has orbifold singularities, and its quantization is given by the Kawasaki Riemann–Roch formula (see Appendix I). From this formula one can deduce that the quantization is quasi-polynomial as a function of α as α varies in a moment cone. Theorem 7.20 is a combinatorial expression of this same fact. For a purely combinatorial proof of Theorem 7.20, see [**Stu**].

Example 7.21. Let $G = S^1$, $\mathbb{Z}_G^* = \mathbb{Z}$, $\alpha_1 = 1$, and $\alpha_2 = 2$. Then

$$\{N(0), N(1), N(2), \dots\} = \{1, 1, 2, 2, 3, 3, \dots\},$$

and

$$N(\alpha) = \frac{\alpha}{2} + \frac{3}{4} + \frac{(-1)^\alpha}{4}.$$

It turns out that $N(\alpha)$ is given by the quasi-polynomial P_C even for a singular value α, on the boundary of the moment cones C:

Theorem 7.22. *For all α in the* closure *of the moment cone C,*

$$N(\alpha) = P_C(\alpha).$$

For a proof, see [**Stu**].

In particular, we have

Corollary 7.23. Let C_1 and C_2 be two different moment cones whose closures contain the singular value $\alpha \in \mathbb{Z}_G^*$. Then

$$P_{C_1}(\alpha) = P_{C_2}(\alpha).$$

Recall that the pre-quantization of a singular reduced space is defined by passing to a nearby regular value; see Section 3.2. If we apply this to $\mathbb{C}^d$ with its standard Kähler form, the dimension of the quantization is equal to $P_C(\alpha)$ where C is the moment cone which contains a nearby regular value that we chose. This fact is a consequence of the Kawasaki Riemann–Roch formula. Corollary 7.23 is a combinatorial expression of the fact that the quantization of a singular reduced space is independent of the choice of desingularization, as long as α remains in the image of the moment map; cf. Chapter 8.

3. The character of $\mathcal{Q}(\mathbb{C}^{\mathbf{d}})$

3.1. Partition functions and the quantization of $\mathbb{C}^{\mathbf{d}}$. Consider the space $\mathbb{C}^d$ with the symplectic form $\omega_r = \sqrt{-1}\sum \epsilon_j dz_j \wedge d\overline{z}_j$ with $\epsilon_1 = \ldots = \epsilon_r = -1$ and $\epsilon_{r+1} = \ldots = \epsilon_d = +1$ as in Section 1. Let a torus G act on it with weights $-\alpha_1, \ldots, -\alpha_d$ as in (7.18) and moment map (7.19) such that $\Phi(0) \in \mathbb{Z}_G^*$. We recall that the quantization $\mathcal{Q}(\mathbb{C}^d, \omega_r)$ was defined by (7.16) as

$$\mathcal{Q}(\mathbb{C}^d, \omega_r) = (-1)^r H^{0,r}(\mathbb{C}^d; \mathbb{L}_r).$$

Thus $(-1)^r \mathcal{Q}(\mathbb{C}^d, \omega_r)$ is a genuine representation of G. In this section we describe this representation in terms of a partition function.

Set

$$\alpha_i^{\#} = \epsilon_i \alpha_i, \qquad i = 1, \ldots, d.$$

Let $N(\cdot)$ be the corresponding partition function, i.e., $N(\alpha)$ is the number of non-negative integer solutions $(k_1, \ldots, k_d)$ to the equation $\sum k_i \alpha_i^{\#} = \alpha$.

PROPOSITION 7.24. *The representation $(-1)^r \mathcal{Q}(\mathbb{C}^d, \omega_r)$ of G splits into a direct sum of one-dimensional weight spaces for G, and the multiplicity with which each weight $\alpha \in \mathbb{Z}_G^*$ occurs is $N(\alpha - \nu)$, where*

$$\nu = \Phi(0) + \sum_{i=1}^{r} \alpha_i^{\#}. \tag{7.31}$$

PROOF. The representation $(-1)^r \mathcal{Q}(\mathbb{C}^d, \omega_r)$ was computed in Theorem 7.11: this is the tensor product of the Bargmann space $\mathcal{L}^2_{\mathrm{hol}}(\overline{\mathbb{C}}^r \times \mathbb{C}^{d-r}; \mu)$ with the one-dimensional space spanned by $s d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_r$. This Hilbert space is the orthogonal sum of the one-dimensional spaces spanned by

$$s \otimes \overline{z}_1^{k_1} \cdots \overline{z}_r^{k_r} z_{r+1}^{k_{r+1}} \cdots z_d^{k_d} d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_r. \tag{7.32}$$

Since the representation of G on the Bargmann space induced by (7.18) transforms the coordinate function z_i according to the weight α_i (see Section 2.5 of Chapter 6), it transforms the vector (7.32) according to the weight

$$\Phi(0) + \sum_{i=1}^{d} k_i \alpha_i^{\#} + \sum_{i=1}^{r} \alpha_i^{\#}. \tag{7.33}$$

Indeed, s transforms according to the weight $\Phi(0)$; for $r+1 \le i \le d$, each $z_i^{k_i}$ transforms according to the weight $k_i \alpha_i = k_i \alpha_i^{\#}$; for $1 \le i \le r$, each $\overline{z}_i^{k_i}$ transforms according to the weight $-k_i \alpha_i = k_i \alpha_i^{\#}$; and, finally, for $1 \le i \le r$, each $d\overline{z}_i$ transforms according to the weight $-\alpha_i = \alpha_i^{\#}$. Adding up these weights, we obtain (7.33).

The multiplicity with which a weight α occurs in the direct sum is equal to the number of ways in which α can be represented as a sum of the form (7.33). This is the number of non-negative integer solutions $(k_1, \ldots, k_d)$ of the equation

$$\sum_{i=1}^{d} k_i \alpha_i^{\#} = \alpha - \Phi(0) - \sum_{i=1}^{r} \alpha_i^{\#}.$$

In other words, this number is $N(\alpha - \nu)$. □

Combining Proposition 7.24 and Corollary 7.15, we obtain

COROLLARY 7.25. The weights $\alpha_i^{\#}$, $i = 1, \ldots, d$ are polarized if and only if each weight $\alpha \in \mathbb{Z}_G^*$ occurs in the representation $(-1)^r \mathcal{Q}(\mathbb{C}^d, \omega_r, \Phi)$ with finite multiplicity.

By Proposition 4.15, the moment map $\Phi(0) + \sum |z_j|^2 \alpha_j^{\#}$ is proper if and only if the weights $\alpha_j^{\#}$ are polarized. Thus, we have

COROLLARY 7.26. The moment map Φ on $(\mathbb{C}^d, \omega_r)$ is proper if and only if each weight $\alpha \in \mathbb{Z}_G^*$ occurs in the representation $\mathcal{Q}(\mathbb{C}^d, \omega_r, \Phi)$ with finite multiplicity.

3.2. The distributional character of $\mathcal{Q}(\mathbb{C}^{\mathbf{d}}, \omega_{\mathbf{r}})$. Denote by τ the representation $(-1)^r \mathcal{Q}(\mathbb{C}^d, \omega_r)$ of G. By Proposition 7.24, an element of G, which we write as $g = \exp \xi$ for $\xi \in \mathfrak{g}$, acts by an infinite diagonal matrix in which the entry $e^{\sqrt{-1}\alpha(\xi)}$ occurs $N(\alpha - \nu)$ times. Taking the trace of this matrix as a formal sum, we conclude that the character of the representation τ is, formally,

$$\chi_\tau(\exp \xi) = \sum_\alpha N(\alpha - \nu) e^{\sqrt{-1}\alpha(\xi)}. \tag{7.34}$$

Furthermore, in the proof of Proposition 7.24 we showed that τ_g acts on each basis element (7.32) of $(-1)^r \mathcal{Q}(\mathbb{C}^d, \omega_r)$ by multiplication by the character

$$e^{\sqrt{-1}\left(\nu + \sum_j k_j \alpha_j^{\#}\right)(\xi)}. \tag{7.35}$$

Adding up these contributions formally, we obtain a formal expression for the character (7.34) as the sum of a geometric progression:

$$\chi_\tau(\exp \xi) = e^{\sqrt{-1}\nu(\xi)} \sum_{k=(k_1,\ldots,k_d)\in\mathbb{Z}_+^d} e^{\sqrt{-1}(k_1\alpha_1^{\#}(\xi) + \ldots + k_d \alpha_d^{\#}(\xi))}. \tag{7.36}$$

This series does not converge as a function on G. However, it does converge as a *distribution*. Therefore, to make the above analysis rigorous, we must make sense of χ_τ as a distributional character. The formula (7.34) will then become the Fourier expansion of χ_τ; see Corollary 7.28 below.

Let us now work out the relevant analysis in this argument. We can think of a representation τ of G on a Hilbert space $\mathcal{H}$ as a function on G that takes values in the space of operators on $\mathcal{H}$. To a smooth function $f \colon G \to \mathbb{C}$ we can then associate the operator τ_f on $\mathcal{H}$ given by

$$\tau_f(v) = \int_G f(g) \tau_g(v)\, dg, \tag{7.37}$$

for $v \in \mathcal{H}$, where dg is the normalized Haar measure.

THEOREM 7.27. *Denote by τ the representation $(-1)^r \mathcal{Q}(\mathbb{C}^d, \omega_r)$ of G, where $\omega_r = \sqrt{-1}\sum \epsilon_i dz_i \wedge d\bar{z}_i$ with $\epsilon_1 = \ldots \epsilon_r = -1$ and $\epsilon_{r+1} = \ldots \epsilon_d = 1$, and where G acts on $\mathbb{C}^d$ with weights $-\alpha_i$, $i = 1, \ldots, d$. Assume that the weights $\alpha_i^{\#} = \epsilon_i \alpha_i$ are polarized. Then for each $f \in C^\infty(G)$ the operator τ_f is of trace class, and the linear functional $\chi_\tau(f) = \operatorname{trace} \tau_f$ on $C^\infty(G)$ is continuous, i.e., χ_τ is a distribution on G. Furthermore,*

$$\chi_\tau(f) = \sum_\alpha \hat{f}(-\alpha) N(\alpha - \nu) \tag{7.38}$$

for $f \in C^\infty(G)$.

PROOF OF THEOREM 7.27. Recall that the Fourier transform $\hat{f}$ of f is the function on the dual group $\mathbb{Z}_G^*$ given by

$$\hat{f}(\alpha) = \int_G f(g) e^{-\sqrt{-1}\alpha}(g)\, dg, \tag{7.39}$$

where the exponential function is viewed as a function on G, i.e., $e^{-\sqrt{-1}\alpha}(g) = e^{-\sqrt{-1}\alpha(\xi)}$ for $g = \exp \xi$, $\xi \in \mathfrak{g}$.

Suppose that the representation τ leaves fixed the one-dimensional space $\mathbb{C} \cdot v$, spanned by a vector v, and that it acts on this space with a weight $-\alpha$, i.e., $\tau_g(v) = e^{-\sqrt{-1}\alpha(\xi)} v$ for all $g = \exp \xi$, $\xi \in \mathfrak{g}$. If we multiply $f(g)$ by this character and integrate over G, we see from (7.37) and (7.39) that v is an eigenvector for τ_f with an eigenvalue $\hat{f}(\alpha)$. Hence, by Proposition 7.24, the operator τ_f is diagonalizable, with eigenvalues $\hat{f}(\alpha)$, $\alpha \in \mathbb{Z}_G^*$, and each eigenvalue $\hat{f}(\alpha)$ occurs with multiplicity $N(-\alpha - \nu)$. To show that τ_f is of trace class we must prove that the sum

$$\sum_\alpha |\hat{f}(\alpha)| N(-\alpha - \nu) \tag{7.40}$$

converges. By Theorem 7.16, since the weights $\alpha_i^\#$ determining the partition function are polarized,

$$|N(-\alpha - \nu)| = O(|\alpha|^{d-n}). \tag{7.41}$$

Since f is C^∞, we also have

$$|\hat{f}(\alpha)| = O(|\alpha|^{-m}) \tag{7.42}$$

for all m. From these two estimates, it follows that (7.40) converges.

The character of τ is then given by (7.38). By the estimates (7.42) and (7.41), this is a continuous linear functional on $C^\infty(G)$. (See, e.g., [**Rudi**].) □

The distribution χ_τ defined in Theorem 7.27 is called the *distributional character* of the representation of G on $(-1)^d Q(\mathbb{C}^d, \omega_r, \Phi)$. Let us now deduce from these results a few properties of χ_τ which we will use in Section 4.

Recall that the Fourier transform of a distribution $\chi\colon C^\infty(G) \to \mathbb{C}$ on G is, by definition, the function on $\mathbb{Z}_G^*$ given by $\hat{\chi}(\alpha) = \chi(e^{-\sqrt{-1}\alpha})$. The following result follows easily from (7.38):

COROLLARY 7.28. The Fourier transform $\hat{\chi}_\tau$ of χ_τ, as a function on $\mathbb{Z}_G^*$, is equal to $\alpha \mapsto N(\alpha - \nu)$.

Let $\eta \in \mathfrak{g}$ be a polarizing vector, so that $\alpha_i^\#(\eta) > 0$ for all i.

PROPOSITION 7.29. *The Fourier transform of χ_τ is supported on the cone*

$$\{\alpha \in \mathbb{Z}_G^* \mid \alpha - \nu \in C(\alpha_1^\#, \dots, \alpha_d^\#)\} \tag{7.43}$$

and, in particular, its support is contained in the half space

$$\alpha(\eta) \geq \nu(\eta). \tag{7.44}$$

PROOF. Since $N(\alpha - \nu)$ is supported on the cone (7.43), this is an immediate consequence of Corollary 7.28. □

Let G_j be the codimension one subgroup of G whose Lie algebra is $\{\alpha_j = 0\}$.

PROPOSITION 7.30. *Assume, as before, that the weights $\alpha_i^\# = \epsilon_i \alpha_i$ are polarized and τ is the representation $(-1)^r \mathcal{Q}(M, \omega_r, \Phi)$ of G. Then the distribution χ_τ is smooth on the complement of the set*

$$\bigcup_{j=1}^{d} G_j,$$

and, on this complement,

$$\chi_\tau(\exp \xi) = (-1)^r e^{\sqrt{-1}\Phi(0)(\xi)} \prod_{j=1}^{d} \left(1 - e^{\sqrt{-1}\alpha_j(\xi)}\right)^{-1}. \tag{7.45}$$

PROOF. By (7.36), the distribution χ_τ is a sum of smooth functions (thought of as distributions). The geometric series (7.36) does not converge as a series of functions. However, (7.36) does converge in the sense of distributions on the complement to the union of the subgroups G_j to

$$\chi_\tau(\exp \xi) = e^{\sqrt{-1}\nu(\xi)} \prod_{j=1}^{d} \left(1 - e^{\sqrt{-1}\alpha_j^\#(\xi)}\right)^{-1}, \tag{7.46}$$

which is a smooth function on this complement.

We will now outline the proof of this for $d = 1$, The proof for an arbitrary dimension d is essentially the same. Let $\alpha = \alpha_1^\#$. We want to show that the distribution

$$\sum_{k=0}^{\infty} e^{\sqrt{-1}k\alpha}$$

converges to the function $(1 - e^{\sqrt{-1}\alpha})^{-1}$ on the set $e^{\sqrt{-1}\alpha} \neq 1$. Let $f \in C^\infty(G)$ be a function supported on this set. Then

$$\frac{f}{1 - e^{\sqrt{-1}\alpha}} - \sum_{k=0}^{N} e^{\sqrt{-1}k\alpha} f = e^{\sqrt{-1}(N+1)\alpha} h,$$

where $h = f(g)(1 - e^{\sqrt{-1}\alpha})^{-1}$ is also a smooth function with support on this set. Integrating this identity over G, we see that the right-hand side is the Fourier coefficient $\hat{h}(-(N+1)\alpha)$ of h which tends to zero as $N \to \infty$.

Recalling that

$$\nu = \Phi(0) + \sum_{j=1}^{r} \alpha_j^\#,$$

we can rewrite the product in (7.46) as

$$\chi_\tau(\exp \xi) = e^{\sqrt{-1}\Phi(0)(\xi)} \prod_{j=1}^{r} \frac{e^{\sqrt{-1}\alpha_j^\#(\xi)}}{1 - e^{\sqrt{-1}\alpha_j^\#(\xi)}} \prod_{j=r+1}^{d} \frac{1}{1 - e^{\sqrt{-1}\alpha_j^\#(\xi)}}$$

or, alternatively, as

$$\chi_\tau(\exp \xi) = (-1)^r e^{\sqrt{-1}\Phi(0)(\xi)} \prod_{j=1}^{r} \frac{1}{1 - e^{-\sqrt{-1}\alpha_j^\#(\xi)}} \prod_{j=r+1}^{d} \frac{1}{1 - e^{\sqrt{-1}\alpha_j^\#(\xi)}}.$$

Since $\alpha_j^\# = -\alpha_j$ when $1 \leq j \leq r$ and $\alpha_j^\# = \alpha_j$ when $r+1 \leq j \leq d$, we obtain (7.45). □

4. A quantum version of the linearization theorem

Let (M, ω, Φ, J) be a compact $2d$-dimensional Hamiltonian G-manifold. (Here G is a torus, ω is a closed G-invariant two-form, Φ is a moment map for ω, and J is an equivariant stable complex structure.) Assume that the fixed points of the G-action are isolated.

Let us recall the assertion of the linearization theorem (see Chapter 4) and remind the reader of all the ingredients that have to go into this theorem for us to be able to quantize it. At each fixed point $p \in M^G$ there exists a G-equivariant complex linear isomorphism

$$(T_pM, J_p) \cong \mathbb{C}^d$$

which converts the isotropy action on T_pM to the action on $\mathbb{C}^d$ given by

$$\tau_p(\exp \xi) = \left(e^{-\sqrt{-1}\alpha_{1,p}(\xi)} z_1, \dots , e^{-\sqrt{-1}\alpha_{d,p}(\xi)} z_d\right),$$

where $-\alpha_{1,p}, \dots , -\alpha_{d,p}$ are the isotropy weights at p. Choose a polarizing vector, i.e., a vector $\eta \in \mathfrak{g}$ such that $\alpha_{k,p}(\eta) \neq 0$ for all k and p, and let $\alpha^{\#}_{k,p} = \epsilon_{k,p}\alpha_{k,p} = \pm\alpha_{k,p}$ be the corresponding polarized weights, so that

$$\alpha^{\#}_{k,p}(\eta) > 0 \tag{7.47}$$

for all k and p. Let $\omega^{\#}_p = \sqrt{-1}\sum \epsilon_{k,p} dz_k \wedge d\overline{z}_k$, and let $\Phi^{\#}_p \colon T_pM \to \mathfrak{g}^*$ be the moment map for this symplectic structure:

$$\Phi^{\#}_p(z) = \Phi(p) + \sum |z_j|^2 \alpha^{\#}_j .$$

The linearization theorem (see Chapter 4) states that

$$(M, \omega, \Phi, J) \sim \bigsqcup_{p \in M^G} (T_pM, \omega^{\#}_p, \Phi^{\#}_p, J_p). \tag{7.48}$$

Recall also that in this cobordism M is oriented and the orientations of T_pM are induced by the orientation of M.

Let us now assume that (M, ω) is pre-quantizable by $(\mathbb{L}, \langle, \rangle, \nabla)$, and that the action of G on M lifts to a pre-quantization action of G on $\mathbb{L}$ by means of the moment map Φ. Then we have the stable complex quantization $\mathcal{Q}(M)$ of M and also have the quantizations of the tangent spaces T_pM described in Section 1.

For compact manifolds, quantization is a cobordism invariant. This follows immediately from the Riemann–Roch formula combined with Stokes' theorem. This argument does not apply directly to the non-compact cobordism (7.48); see, however, [**Brav4**]. Nevertheless, the result is still true:

THEOREM 7.31 (The Quantum Linearization Theorem). *As virtual unitary G-representations,*

$$\mathcal{Q}(M, \omega, \Phi, J) = \sum_p \mathcal{Q}(T_pM, \omega^{\#}_p, \Phi^{\#}_p, J_p). \tag{7.49}$$

REMARK 7.32. The equality (7.49) is interpreted in the Gröthendieck group of formal differences of unitary representations of G with trace-class characters. Alternatively, Theorem 7.31 can be understood as equalities of multiplicities: for all $\alpha \in \mathbb{Z}^*_G$,

$$\text{mult}(\alpha) = \sum \text{mult}_p(\alpha), \tag{7.50}$$

where $\mathrm{mult}(\alpha)$ is the multiplicity with which α occurs in the virtual representation $\mathcal{Q}(M, \omega, \Phi, J)$ and $\mathrm{mult}_p(\alpha)$ is the multiplicity with which α occurs in the signed representation $\mathcal{Q}(T_pM, \omega_p^\#, \Phi_p^\#, J_p)$. Since the weights $\alpha_i^\#$ are polarized, the multiplicities are finite by Corollary 7.25.

The multiplicity formula (7.50), when applied to a coadjoint orbit for a compact Lie group, yields the famous *Kostant multiplicity formula*. Formula (7.50) can also be found, in a slightly different guise, in [**GLS**].

Theorem 7.31 is equivalent to the distributional identity

$$\chi_M = \sum \chi_p \tag{7.51}$$

where χ_M is the character of $\mathcal{Q}(M)$ and where χ_p is the distributional character of $\mathcal{Q}(T_pM)$. We note that this equality is slightly stronger than the Atiyah–Bott Lefschetz theorem: on an open dense subset of G, the distributions χ_p are given by smooth functions, and on this set, (7.51) becomes (6.41).

REMARK 7.33. An important special case of Theorem 7.31 is the following result of Bernstein–Gelfand–Gelfand, [**BGG**]. Let K be a compact semi-simple Lie group having G as its maximal torus and let M be an integral coadjoint orbit of K. By a theorem of Bott–Borel–Weil–Kostant, the quantization representation ρ of K on $\mathcal{Q}(M)$ is irreducible and, in fact, all irreducible representations of K can be obtained this way. (See Example 6.39.) The theorem of Bernstein–Gelfand–Gelfand asserts that the representation ρ can be decomposed into a virtual sum of representations which, as G modules, have a much simpler structure than the G module $\mathcal{Q}(M)$ itself. These "Verma modules" are just the summands on the right-hand side of (7.49).

PROOF OF THEOREM 7.31. Without loss of generality, we may assume that the orientation of M is induced by J, and hence the tangent spaces T_pM carry complex orientations induced by J_p. (The reversal of the orientation of M results in that both the right- and left-hand sides of (7.49) change sign; see Remark 7.10.)

Let χ_M be the character of the representation of G on $\mathcal{Q}(M, \omega, \Phi, J)$. For each fixed point p, let χ_p be the character of the virtual representation of G on $\mathcal{Q}(T_pM, \omega_p^\#, \Phi_p^\#, J_p)$. By Theorem 7.27 and, in particular, by (7.38), χ_p determines the multiplicities $\mathrm{mult}_p(\alpha)$ and hence the representation $\mathcal{Q}(T_pM, \omega_p^\#, \Phi_p^\#, J_p)$. Since $\mathcal{Q}(M, \omega, \Phi, J)$ is finite–dimensional, it is also determined by its character. Therefore, it suffices to prove that

$$\chi_M = \sum \chi_p. \tag{7.52}$$

The summands in (7.52) were computed in Section 3. Indeed, for each p, the tangent space $(T_pM, \omega_p^\#, \Phi_p^\#, J_p)$ is isomorphic to $\mathbb{C}^d$, equipped with the G-action with the weights $\alpha_{k,p}$, the symplectic form $(\sqrt{-1})^d \sum_j \epsilon_{k,p} dz_j \wedge d\overline{z}_j$, and the moment map

$$\Phi_p^\#(z) = \Phi(p) + \sum_j \epsilon_{k,p} |z_j|^2 \alpha_{k,p} = \Phi(p) + \sum_j |z_j|^2 \alpha_{k,p}^\#,$$

where the signs $\epsilon_{k,p}$ are such that the weights $\alpha_{k,p}^\#$ are polarized. By Proposition 7.30,

$$\chi_p(\exp \xi) = e^{\sqrt{-1}\Phi(p)(\xi)} \prod \left(1 - e^{\sqrt{-1}\alpha_{k,p}(\xi)}\right)^{-1}$$

on the set G_0 where $e^{\sqrt{-1}\alpha_{k,p}(\xi)} \neq 1$ for all $p \in M^G$ and $k = 1, \dots, d$. By the Atiyah–Bott Lefschetz fixed point formula (see (6.41)), we conclude that

$$\chi_M(\exp \xi) = \sum_p \chi_p(\exp \xi)$$

on the set G_0. In other words, the distribution

$$\chi = \Big(\sum \chi_p\Big) - \chi_M$$

is supported on the complement of G_0, i.e., on the union of the sets

$$e^{\sqrt{-1}\alpha_{k,p}(\xi)} = 1, \quad k = 1, \dots, d, \quad p \in M^G.$$

This alone is not sufficient to establish the identity (7.52). However, we have some additional information about χ. Namely, we know, by Proposition 7.29, that for some constant C_0 the Fourier transform $\hat{\chi}$ is supported on the half-space

$$\alpha(\eta) \geq C_0, \quad \alpha \in \mathbb{Z}_G^*. \tag{7.53}$$

We will show that this forces (7.52) to hold identically by proving the following result. □

LEMMA 7.34. *Let $\alpha_j^\#$, $j = 1, \dots, l$, be a collection of weights satisfying $\alpha_j^\#(\eta) > 0$, and let χ be a distribution on G. Assume that χ is supported on the union of the sets*

$$\{\exp(\xi) \mid e^{\sqrt{-1}\alpha_j^\#(\xi)} = 1, \quad j = 1, \dots, l\}, \tag{7.54}$$

and that the Fourier transform of χ is supported on the half-space (7.53). Then $\chi \equiv 0$.

PROOF. Since G is compact, χ is of the form $P\nu$, where ν is a *measure* with support on the set (7.54) and P a differential operator of *finite* order (see [**Rudi**]). Hence there exist non-negative integers, $N_1, \dots, N_l$ such that

$$(1 - e^{\sqrt{-1}\alpha_1^\#})^{N_1} \dots (1 - e^{\sqrt{-1}\alpha_l^\#})^{N_l} \chi \equiv 0. \tag{7.55}$$

Without loss of generality, we can assume that these integers have been chosen to be as *small as possible*; *i.e.*, if $N_j > 0$ and we replace N_j by $N_j - 1$, the identity (7.55) does not hold. We claim that in this case the N_j's all have to be zero and, hence, $\chi \equiv 0$.

To prove the claim, we argue by contradiction. First observe that by taking the Fourier transform of (7.55) we obtain the identity

$$(1 - T_{\alpha_1^\#})^{N_1} \dots (1 - T_{\alpha_l^\#})^{N_l} \widehat{\chi} = 0,$$

where $T_{\alpha_j^\#}$ is the operator which transforms a function $f \colon \mathbb{Z}_G^* \to \mathbb{C}$ to the function

$$T_{\alpha_i} f(\mu) = f(\mu - \alpha_j^\#).$$

In particular, if f is supported on the set (7.53), $T_{\alpha_j^\#} f$ is supported on the set

$$\alpha(\eta) \geq C_0 + \alpha_j^\#(\eta).$$

Suppose now that $N_r > 0$. Let

$$h = (1 - T_{\alpha_1^\#})^{N_1} \dots (1 - T_{\alpha_r^\#})^{N_r - 1} \dots (1 - T_{\alpha_l^\#})^{N_l} \widehat{\chi}.$$

Then h is supported on the set $\alpha(\eta) \geq C_0'$ with

$$C_0' = C_0 + N_1\alpha_1^{\#}(\eta) + \ldots + (N_r - 1)\alpha_r^{\#}(\eta) + \ldots + N_l\alpha_l^{\#}(\eta).$$

Moreover,

$$(1 - T_{\alpha_r^{\#}})h = 0,$$

and thus

$$h = T_{\alpha_r^{\#}}h = T^2_{\alpha_r^{\#}}h = \ldots = T^N_{\alpha_r^{\#}}h$$

for an arbitrary $N > 0$. Therefore, h is supported on the set

$$\alpha(\eta) \geq C_0' + N\alpha_r^{\#}(\eta).$$

Hence, since N is arbitrary, $h \equiv 0$. Thus (7.55) holds with $N_1, \ldots, N_l$ replaced by $N_1, \ldots, N_r - 1, \ldots, N_l$. This completes the proof of Lemma 7.34 and the proof of Theorem 7.31. □