CHAPTER 4

The linearization theorem

The simplest group actions on manifolds are linear representations on vector spaces. These are not legitimate objects for ordinary equivariant cobordisms, because they are not compact. However, they are legitimate objects for *proper* cobordisms. In this chapter we will give the linearization theorem, which, roughly, "decomposes" a G-manifold, when G is a torus, into linear G-representations.

For a torus action with isolated fixed points on a compact manifold, this theorem can be informally read as

$$M \sim \bigsqcup_{p \in M^G} T_p M,$$

where M^G is the fixed point set and T_pM is the tangent space to M at p. For non-isolated fixed points, on the right-hand side we get the normal bundles to the connected components of M^G.

The linearization theorem holds for manifolds with η-polarized (abstract, or Hamiltonian) moment maps, for any pre-chosen $\eta \in \mathfrak{g}$. It implies that the η-polarized cobordism classes are generated, as a group, by vector spaces and vector bundles.

REMARK 4.1. Essentially the same is true with *proper* (not polarized) cobordisms: a proper moment map can be "cut" (à la Lerman, [**Ler1**]) so that each resulting piece is polarized with respect to some (not the same) η; the linearization theorem can then be applied to the pieces. See Section 8 of Chapter 5.

1. The simplest case of the linearization theorem

To capture the main idea of the linearization theorem, let us concentrate on the moment map and ignore the two-form. The notion of a "moment map without a two-form" was made precise in Chapter 3, where we introduced *abstract moment maps*. For an action of the circle group S^1, an abstract moment map is simply a real valued S^1-invariant function which is locally constant on the fixed point set; it is *polarized* if it is proper and bounded from below.

PROPOSITION 4.2. *Let $G = S^1$ be the circle group. Let M be a manifold with a G-action with isolated fixed points, and let $\Psi \colon M \to \mathbb{R}$ be a polarized abstract moment map. Then there exists a cobordism of manifolds with G-actions and polarized abstract moment maps,*

$$(M, \Psi) \sim \bigsqcup_{p \in M^G} (T_pM, \Psi_p^{\#}), \tag{4.1}$$

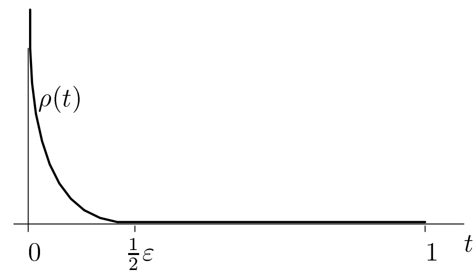

FIGURE 4.1. A proper function on (0,1]

where for each fixed point $p \in M^G$, the vector space T_pM is equipped with the linear isotropy G-action induced from M, and $\Psi_p^\#\colon T_pM \to \mathbb{R}$ is a polarized abstract moment map with $\Psi_p^\#(0) = \Psi(p)$.

REMARK 4.3. The map $\Psi_p^\#$ on the right-hand side of (4.1) is not specified. However, it is unique up to proper cobordism. Indeed, two such maps are cobordant through the trivial cobordism $[0,1] \times T_pM$ with the polarized abstract moment map $\tilde{\Psi}(t,v) = (1-t)(\Psi_p^\#)_1(v) + t(\Psi_p^\#)_2(v)$. (See also Section 4 of Chapter 3.)

PROOF OF PROPOSITION 4.2. Consider the product $(0,1] \times M$, with the circle action induced by that on M. This provides a non-compact cobordism of M with the empty set. The function $(t,m) \mapsto \Psi(m)$ is an abstract moment map, but it is not proper.

We correct this by adding a function $\rho(t)$ that approaches ∞ as t approaches 0, and that vanishes for $\frac{1}{2}\epsilon \le t \le 1$, for some $0 < \epsilon < 1$. See Figure 4.1.

The sum

$$\tilde{\Psi}(t,m) = \Psi(m) + \rho(t), \qquad (t,m) \in (0,1] \times M,$$

is polarized. This follows from the fact that $\rho(t)$ is polarized on $(0,1]$ and $\Psi(m)$ is polarized on M. If the fixed point set is empty, the function $\tilde{\Psi}$ on the product $(0,1] \times M$ provides a proper cobordism of (M,Ψ) with the empty set. This finishes the proof.

If the fixed point set is non-empty, the function $\tilde{\Psi}$ is polarized, but it is not an abstract moment map. The reason is that the components of the fixed point set in $(0,1] \times M$ are the sets $(0,1] \times \{p\}$, where p is a fixed point in M, and the function $\tilde{\Psi}$ is not constant on these components. However, in this case we can take the same function $\tilde{\Psi}$ on the manifold

$$W = ((0,1] \times M) \smallsetminus \bigsqcup_{p \in M^{S^1}} B_p,$$

where B_p is the set of points in $(0,1] \times M$ which are ϵ-close to the point $(0,p)$, i.e.,

$$B_p = \left\{ (t,m) \in (0,1] \times M \;\mid\; t^2 + (\text{distance}(m,p))^2 < \epsilon^2 \right\},$$

with respect to some metric on M. Here ϵ is the same as in the definition of the function $\rho(t)$. Consider the fixed point set W^G. The following observation is crucial:

If $(t,p) \in W^G$, then $\rho(\cdot)$ vanishes on a neighborhood of t.

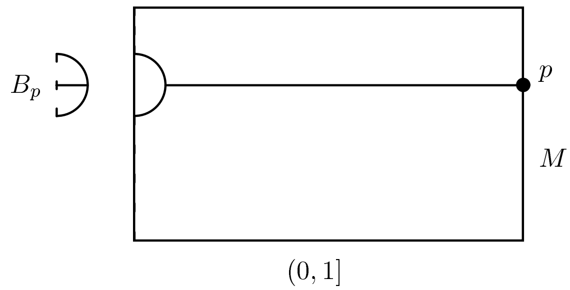

FIGURE 4.2. Proof of the linearization theorem

For if $\rho(t) \neq 0$, then $t < \epsilon$, which implies that $(t,p) \in B_p$ if p is a fixed point. It follows that the restriction of $\tilde{\Psi}$ to W is an abstract moment map. This map is polarized because it is the restriction of the polarized function $\tilde{\Psi}$ to the closed subset W of $(0,1] \times M$.

The removal of B_p creates a boundary component of W that is equivariantly diffeomorphic to the ϵ-neighborhood of p in M via the projection map $(t,m) \mapsto m$. We may assume that the metric is chosen so that this ϵ-neighborhood is equivariantly diffeomorphic to the tangent space T_pM and the closures of these neighborhoods are disjoint from each other for different p's. Then there exists an equivariant diffeomorphism

$$\partial W \cong \bigsqcup_{p \in M^G} T_pM \sqcup M. \tag{4.2}$$

The pullback of $\tilde{\Psi}$ by the inclusion map $T_pM \to \partial W$ is a polarized abstract moment map $\Psi_p^{\#} \colon T_pM \to \mathbb{R}$ such that $\Psi_p^{\#}(0) = \Psi(p)$ and $\Psi_p^{\#}(v) \to \infty$.

The pair $(W, \tilde{\Psi})$ is a cobordism with the desired properties. □

2. The Hamiltonian linearization theorem

Let G be a torus, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space. Recall that a map to $\mathfrak{g}^*$ is *η-polarized*, for $\eta \in \mathfrak{g}$, if its η-component is proper and bounded from below. A map from a *compact* space to $\mathfrak{g}^*$ is automatically η-polarized for all η.

Let G act on a manifold M. For each Lie algebra element $\eta \in \mathfrak{g}$, let $M^\eta = \{\eta_M = 0\}$ denote the zero set of the corresponding vector field. For a generic η, this set coincides with the fixed point set M^G of G. Indeed, $M^\eta \neq M^G$ only if η belongs to the infinitesimal stabilizer of a point in $M \smallsetminus M^G$, and these infinitesimal stabilizers form a countable union of proper subspaces of $\mathfrak{g}$ (see Corollary B.46). For *any* η, the connected components of the set M^η are closed submanifolds F of M. (See Corollary B.40.) Note that these components are compact if Φ is η-polarized (because $\Phi^\eta|_F$ is both proper and constant). For each component F of M^η let NF denotes its normal bundle in M. The group G acts on NF by bundle automorphisms; this action is induced from the G-action on M. The total space of NF is oriented; the orientation is induced from the orientation on M. We can now state the Hamiltonian linearization theorem.

THEOREM 4.4 (Hamiltonian linearization theorem). *Fix a torus G and an element $\eta \in \mathfrak{g}$ of its Lie algebra. Let (M, ω, Φ) be a Hamiltonian G-manifold whose* *moment map Φ is η-polarized. Then there exists an η-polarized Hamiltonian cobordism*

$$(M, \omega, \Phi) \sim \bigsqcup_{F \in \pi_0(M^\eta)} (NF, \omega_F^\#, \Phi_F^\#), \tag{4.3}$$

where for each connected component F of M^η, its normal bundle NF is equipped with an invariant closed two-form $\omega_F^\#$ and an η-polarized moment map $\Phi_F^\#$, and the pullbacks of $\omega_F^\#$ and $\Phi_F^\#$ to the zero section coincide with the pullbacks of ω and Φ to F.

We stress that the two-forms $\omega_F^\#$ are not induced from the manifold. Even in the case that the two-form ω is identically zero, the linearization theorem is interesting and non-trivial.

Proof of Theorem 4.4. Choose an invariant Riemannian metric on M and an $0 < \epsilon < 1$, such that the ϵ-neighborhood of each connected component F of M^η is equivariantly diffeomorphic to the normal bundle NF and the closures of these neighborhoods are disjoint for different F's. (The exponential map identifies a neighborhood of F with a disc bundle in NF, which, in turn, is isomorphic to all of NF.) Let B_F be the set of points in $(0,1] \times M$ which are ϵ-close to $\{0\} \times F$. Let

$$W = ((0,1] \times M) \smallsetminus \bigsqcup_F B_F, \tag{4.4}$$

so that

$$\partial W = \bigsqcup_F \partial B_F \sqcup (\{1\} \times M) \cong \bigsqcup_F N_F \sqcup M. \tag{4.5}$$

Let $\rho\colon (0,1] \to \mathbb{R}$ be a function such that $\rho(t)$ approaches infinity as t approaches 0 and vanishes for $\frac{1}{2}\epsilon \leq t \leq 1$. Note that,

$$\text{if } \eta_W = 0 \text{ at } (t,m)\text{, then } \rho = 0 \text{ on a neighborhood of } t.$$

This guarantees that the one-form β on W defined by

$$\beta(\,\cdot\,) = \begin{cases} \rho(t) \dfrac{\langle \cdot\,, \eta_W \rangle}{\langle \eta_W, \eta_W \rangle} & \eta_M \neq 0, \\ 0 & \eta_M = 0 \end{cases}$$

is smooth. We take the two-form and moment map on W given by

$$\tilde{\omega} = \pi_M^* \omega - d\beta \qquad \text{and} \qquad \tilde{\Phi}^\xi = \pi_M^* \Phi^\xi + \beta(\xi_W), \tag{4.6}$$

where π_M is the projection to M. It is easy to verify the moment map equation $d\tilde{\Phi}^\xi = \iota(\xi_W)\tilde{\omega}$.

Substituting $\xi = \eta$ in (4.6), we see that the η-component of the cobording moment map is

$$\tilde{\Phi}^\eta(t,m) = \Phi^\eta(m) + \rho(t). \tag{4.7}$$

Because $\rho(\cdot)$ is polarized (proper and bounded from below) on $(0,1]$ and $\Phi^\eta(\cdot)$ is polarized on M, the sum (4.7) is polarized on $(0,1] \times M$, and hence on the closed subset W. Therefore, $(W, \tilde{\omega}, \tilde{\Phi})$ is an η-polarized cobordism.

Because $\beta = 0$ near $\{1\} \times M$, the pullbacks of $\tilde{\omega}$ and $\tilde{\Phi}$ to M by the inclusion map $M \cong \{1\} \times M \to \partial W$ are equal to ω and Φ.

For each component F of M^η, the normal bundle NF embeds into ∂W as the boundary component ∂B_F, and the zero section $F \subset NF$ embeds into ∂W as the set $\{\epsilon\} \times F$. Because $\beta = 0$ near this set, the pullbacks of $\tilde{\omega}$ and $\tilde{\Phi}$ to F via the inclusions $F \to NF \to \partial W$ are equal to the pullbacks of ω and Φ via the inclusion $F \to M$.

The Hamiltonian G-manifold $(W, \tilde{\omega}, \tilde{\Phi})$ is an η-polarized cobordism with the required properties. □

REMARK 4.5. For future reference we note that the boundary piece $M \subseteq \partial W$ is a strong deformation retract of W and that $TW = \pi_M^* TM \oplus \mathbb{R}$ where $\pi_M \colon W \to M$ is the projection. Moreover, inclusion map $i_F \colon NF \to \partial W$ can be chosen so that the composition $\pi_M \circ i_F \colon NF \to M$, when differentiated along F in the fiber directions, induces the identity map on NF.

The two-form and moment map on the right-hand side of (4.3) are not specified. However, they are unique up to cobordism by the following lemma:

LEMMA 4.6. *Let $\pi \colon E \to F$ be a vector bundle and let a torus act on it by bundle automorphisms. Let ω_0 and ω_1 be closed invariant two-forms on the total space of E, with moment maps Φ_0 and Φ_1 that are η-polarized, such that $i_F^*\omega_0 = i_F^*\omega_1$ and $i_F^*\Phi_0 = i_F^*\Phi_1$, where $i_F \colon F \to E$ is the inclusion of F into E as the zero section. Then the trivial cobordism $[0,1] \times E$ carries a two-form and an η-polarized moment map which gives rise to a cobordism between (E, ω_0, Φ_0) and (E, ω_1, Φ_1).*

We also note the following special case:

LEMMA 4.7. *Let V be a vector space with a linear torus action. Let ω_0 and ω_1 be closed invariant two-forms with moment maps Φ_0 and Φ_1 which are η-polarized and which take the same value at the origin. Then the trivial cobordism $[0,1] \times V$ carries a two-form and an η-polarized moment map giving a cobordism connecting (V, ω_0, Φ_0) and (V, ω_1, Φ_1).*

PROOF OF LEMMA 4.6. Because i_F is a homotopy equivalence and $i_F^*\omega_0 = i_F^*\omega_1$, the forms ω_0 and ω_1 are in the same cohomology class. Let β be a one-form such that $\omega_1 = \omega_0 - d\beta$. Without loss of generality, we may assume that $i_F^*\beta$ vanishes. (Otherwise, we replace β by $\beta - \pi^* i_F^* \beta$.) Furthermore, by the averaging argument, we can assume that β is invariant. The sum $\Phi_0^\xi + \beta(\xi_E)$ is a moment map for ω_1 which coincides with Φ_0, and therefore with Φ_1, along F. On the trivial cobordism $[0,1] \times E$, we take the two-form $\omega_0 - d(t\beta)$ and moment map $\Phi_0^\xi + t\beta(\xi_E) = (1-t)\Phi_0^\xi + t\Phi_1^\xi$. □

REMARK 4.8. If M^η consists of isolated fixed points (which implies $M^\eta = M^G$), equation (4.3) turns into

$$(M, \omega, \Phi) \sim \bigsqcup_{p \in M^G} (T_pM, \omega_p^\#, \Phi_p^\#). \tag{4.8}$$

If ω is symplectic, each T_pM is a symplectic vector space, with the two-form $\omega_p = \omega|_{T_pM}$ induced from M. However, the corresponding quadratic moment map, which is equal to the Hessian of Φ at p, might not be polarized. Still, we can choose the forms $\omega_p^\#$ to be symplectic, although different from the forms ω_p. See Proposition 4.18.

In contrast with this, when there are non-isolated fixed points, the forms $\omega_F^\#$ *cannot* be in general chosen symplectic. For example, if the Duistermaat–Heckman measure for $(NF, \omega_F^\#, \Phi_F^\#)$ is positive in one region and negative in another, it cannot arise as a push-forward of a symplectic measure. See Example 4.19.

Various structures on the manifold M naturally extend to the linearization theorem cobordism (4.3). Let W denote the cobording manifold (4.4) and $\pi_M : W \to M$ the projection map.

REMARK 4.9. (1) Let $\mathbb{L} \to M$ be an equivariant complex line bundle. This bundle pulls back to an equivariant complex line bundle $\mathbb{L}|_F$ on F, and further pulls back to an equivariant complex line bundle $\mathbb{L}_F$ on the normal bundle NF. Then

$$\mathbb{L} \sim \bigsqcup_F \mathbb{L}_F$$

via the line bundle $\pi_M^* \mathbb{L}$ on the cobording manifold W. This argument applies also to vector bundles, principal bundles, etc.

(2) Let $[\beta]$ be an equivariant cohomology class on M. (See Appendix C for the relevant definitions.) This class pulls back to an equivariant cohomology class $[\beta|_F]$ on F, and further pulls back to an equivariant cohomology class $[\beta_F]$ on the normal bundle NF. Then

$$[\beta] \sim \bigsqcup_F [\beta_F]$$

via the equivariant cohomology class $\pi_M^*[\beta]$ on the cobording manifold W.

(3) Recall that an equivariant stable complex structure on M is a fiberwise complex structure on $TM \oplus \mathbb{R}^k$ for some k. For each connected component F of M^η, for any $\eta \in \mathfrak{g}$, the normal bundle NF inherits a stable complex structure J_F. Then

$$\bigsqcup_F J_F \sim J$$

via the stable complex structure $\tilde{J} = \pi_M^* J \oplus \sqrt{-1}$ on $TW \oplus \mathbb{R} = \pi_M^* TM \oplus \mathbb{R}^2 \cong \pi_M^* TM \oplus \mathbb{C}$. For more details, see Section 1.3 of Appendix D.

We will explicitly state two special cases. The first of these is the linearization theorem for stable complex Hamiltonian G-manifolds with isolated fixed points. We will use it in our cobordism proof of "quantization commutes with reduction", in Chapter 8.

THEOREM 4.10. *Let G be a torus and (M, ω, Φ) a compact Hamiltonian G-manifold. Suppose that G acts with isolated fixed points. Let $\eta \in \mathfrak{g}$ be a generic Lie algebra element, (generic in the sense that $M^\eta = M^G$). Let J be an equivariant stable complex structure on M. For each fixed point $p \in M^G$, consider the tangent space T_pM with the orientation, linear isotropy G-action, and complex structure J_p that are induced from M. Then there exists an η-polarized Hamiltonian complex cobordism,*

$$(M, \omega, \Phi, J) \sim \bigsqcup_{p \in M^G} (T_pM, \omega_p^\#, \Phi_p^\#, J_p), \tag{4.9}$$

where $\omega_p^\#$ and $\Phi_p^\#$ are a two-form and moment map on T_pM such that $\Phi_p^\#(0) = \Phi(p)$.

The second special case is the linearization theorem with equivariant cohomology classes. We will use it in our topological version of the Jeffrey-Kirwan theorem, in Section 7 of Chapter 5.

THEOREM 4.11. *Let (M, ω, Φ) be a Hamiltonian G-manifold with isolated fixed points. For each equivariant cohomology class $[\beta] \in H_G^*(M)$, there exists an η-polarized equivariant cobordism*

$$(M, \omega, \Phi, [\beta]) \sim \bigsqcup_{p \in M^G} (T_p M, \omega_p^{\#}, \Phi_p^{\#}, [\beta(p)]).$$

3. The linearization theorem for abstract moment maps

In some contexts it is useful to work with a variant of the linearization theorem which involves abstract moment maps instead of ordinary moment maps. (See [**Ka3**].)

THEOREM 4.12 (Linearization theorem for abstract moment maps). *Let a torus G act on a manifold M and let $\Psi\colon M \to \mathfrak{g}^*$ be an η-polarized abstract moment map, for some $\eta \in \mathfrak{g}$. For each component F of the zero set $M^\eta = \{\eta_M = 0\}$, equip its normal bundle NF with the G-action induced from M. Then there exists an η-polarized cobordism*

$$(M, \Psi) \sim \bigsqcup_{F \in \pi_0(M^\eta)} (NF, \Psi_F^{\#}), \tag{4.10}$$

where $\Psi_F^{\#}$ is an η-polarized abstract moment map whose restriction to the zero section coincides with the restriction of Ψ to F.

In addition,

(1) *If M is oriented and NF is equipped with the orientation induced from M,* (4.10) *is an oriented cobordism.*
(2) *If $\mathbb{L} \to M$ is an equivariant complex line bundle and $\mathbb{L}_F \to NF$ is the induced equivariant complex line bundle on NF, the cobordism* (4.10) *carries a line bundle whose restrictions to M and the components NF are $\mathbb{L}$ and $\mathbb{L}_F$, respectively.*
(3) *If $[\beta]$ is an equivariant cohomology class on M and $[\beta_F]$ is the induced equivariant cohomology class on NF, the cobordism* (4.10) *carries an equivariant cohomology class whose restrictions to M and the components NF are $[\beta]$ and $[\beta_F]$, respectively.*
(4) *If J is an equivariant stable complex structure on M and J_F is the induced equivariant stable complex structure on NF, the cobordism* (4.10) *carries an equivariant stable complex structure whose restrictions to M and the components NF are J and J_F, respectively.*

REMARK 4.13. The abstract moment map $\Psi_F^{\#}$ on the right-hand side of (4.10) is unique up to cobordism. Indeed, suppose that $\Psi_r\colon NF \to \mathfrak{g}^*$, for $r = 0, 1$, are both η-polarized and coincide on the zero section. Then they are cobordant through the trivial cobordism $[0,1] \times NF$ with the η-polarized abstract moment map $(1-t)\Psi_0 + t\Psi_1$.

PROOF OF THEOREM 4.12. Choose an invariant Riemannian metric on M and an $0 < \epsilon < 1$ such that the ϵ-neighborhood of each connected component F of M^η is equivariantly diffeomorphic to the normal bundle NF and such that the closures of these neighborhoods are disjoint for different components F. Let B_F be the set of points in $(0,1] \times M$ that are ϵ-close to $\{0\} \times F$. Then $W := ((0,1] \times M) \smallsetminus \bigsqcup_F B_F$ provides a non-compact equivariant cobordism between M and $\bigsqcup_F NF$.

Let $\rho\colon (0,1] \to \mathbb{R}$ be a function such that $\rho(t)$ approaches infinity as t approaches 0 and vanishes for $\frac{1}{2}\epsilon \le t \le 1$.

For any $m \in M \smallsetminus M^\eta$, let $\alpha_m \in \mathfrak{g}^*$ be such that $\alpha_m(\eta) = 1$ and $\alpha_m(\xi) = 0$ for all ξ in the infinitesimal stabilizer $\mathfrak{g}_m = \{\xi \mid \xi_M(p) = 0\}$.

For each $(t,m) \in W \smallsetminus W^\eta$, let $U_{t,m}$ be an open invariant neighborhood which retracts to the orbit of (t,m). The constant function α_m is an abstract moment map on this neighborhood.

Let $\varphi_i(t,m)$ be an invariant partition of unity on $W \smallsetminus W^\eta$, with the support of φ_i contained in the open set U_{t_i,m_i}. Define

$$\widetilde{\Psi}(t,m) = \Psi(m) + \rho(t) \sum_i \varphi_i(t,m)\alpha_{m_i}.$$

Because $\rho(\cdot)$ vanishes on a neighborhood of W^η, the function $\widetilde{\Psi}$ is well defined and smooth on W. It is not hard to check that $\widetilde{\Psi}$ is an abstract moment map and is η-polarized. This function restricts to Ψ on M and to $\Psi_F^\#$ on NF. An orientation, line bundle, equivariant cohomology class, and/or stable complex structure on M naturally extend to $M \times (0,1]$ and restrict to the subset W, exactly as in Remark 4.9. □

4. Linear torus actions

The right-hand side of the linearization theorem identity is comprised of linear representations of tori. Let us recall standard facts about such representations. We refer the reader to [**Ad**] for proofs of these and to Appendix A for our conventions on weights.

Every complex linear representation of a torus G splits into complex one–dimensional representations, and these are parametrized by the elements of the weight lattice $\mathbb{Z}_G^*$. So every complex linear representation of G is equivalent to the following representation of G on $\mathbb{C}^d$:

$$\exp\xi\colon (z_1, \dots, z_d) \mapsto (e^{-\sqrt{-1}\alpha_1(\xi)} z_1, \dots, e^{-\sqrt{-1}\alpha_d(\xi)} z_d), \quad \xi \in \mathfrak{g}. \tag{4.11}$$

The weights for the representation, $-\alpha_1, \dots, -\alpha_d$ are unique up to permutation.

A representation of G on a *real* vector space is isomorphic to a direct sum of a trivial representation and a representation of the form (4.11) with all $\alpha_j \neq 0$. The weights α_j are now determined up to permutation and sign: the one-dimensional representations of G on $\mathbb{C}$ with weights α_j and $-\alpha_j$ are isomorphic through $z \mapsto \overline{z}$. The elements $\pm\alpha_j \in (\mathbb{Z}_G^* \smallsetminus \{0\})/\pm 1$ are called the *real weights* of the representation; see [**Ad**].

Symplectic G-representations are similar to complex representations: any symplectic vector space with a linear symplectic G-action is isomorphic to $\mathbb{C}^d$ with the standard symplectic form and with the action (4.11).

The moment map for the action (4.11) is

$$\Phi(z) = \Phi(0) + \frac{1}{2}\sum_{j=1}^{d} |z_j|^2 \alpha_j. \tag{4.12}$$

This is most easily seen in polar coordinates, where the standard symplectic form is

$$\omega = \sum r_j dr_j \wedge d\theta_j,$$

the generating vector fields are

$$\xi_M = -\sum \alpha_j(\xi)\frac{\partial}{\partial\theta_j}, \qquad \xi \in \mathfrak{g}.$$

Hence, the moment map

$$\Phi(z) = \Phi(0) + \frac{1}{2}\sum r_j^2\alpha_j$$

satisfies

$$d\Phi^\xi = \sum (r_j dr_j)\,\alpha_j(\xi).$$

In the rest of this section we assume that

$$\Phi(0) = 0.$$

The image of the moment map is the convex polyhedral cone

$$C(\alpha_1,\dots,\alpha_d) = \left\{\sum s_j\alpha_j \mid s \in \mathbb{R}^d_+\right\}.$$

The moment map itself is the composition of the map

$$J\colon \mathbb{C}^d \to \mathbb{R}^d_+, \qquad J(z_1,\dots,z_d) = \frac{1}{2}\left(|z_1|^2,\dots,|z_d|^2\right) \tag{4.13}$$

with the map

$$\pi\colon \mathbb{R}^d_+ \to \mathfrak{g}^*, \qquad \pi(s) = \sum s_j\alpha_j\,, \tag{4.14}$$

where $\mathbb{R}^d_+$ is the positive orthant

$$\mathbb{R}^d_+ = \{s \in \mathbb{R}^d \mid s_j \geq 0, j = 1,\dots,d\}.$$

Notice that the G-action (4.11) is given by the homomorphism

$$G \to (S^1)^d, \qquad \exp\xi \mapsto (e^{i\alpha_1(\xi)},\dots,e^{i\alpha_d(\xi)}), \tag{4.15}$$

followed by the action of $(S^1)^d$ on $\mathbb{C}^d$ by

$$(a_1,\dots,a_d)\colon (z_1,\dots,z_d) \mapsto (a_1^{-1}z_1,\dots a_d^{-1}z_d).$$

The map (4.13) is the moment map for this $(S^1)^d$ action, and the projection map

$$\mathbb{R}^d \to \mathfrak{g}^*, \qquad s \mapsto \sum s_j\alpha_j$$

in (4.14) is the dual to the inclusion (4.15).

For $\alpha \in C(\alpha_1,\dots,\alpha_d)$ we consider the polytope

$$\Delta_\alpha := \{s \in \mathbb{R}^d_+ \mid \sum s_j\alpha_j = \alpha\}. \tag{4.16}$$

This polytope is the intersection of the positive orthant $\mathbb{R}^d_+$ with the affine subspace

$$A(\alpha) = \{x \in \mathbb{R}^d \mid \sum x_j\alpha_j = \alpha\}.$$

Equivalently, the polytope Δ_α is a level set of the map (4.14):

$$\Delta_\alpha = \pi^{-1}(\alpha).$$

We say that the elements α_j of $\mathfrak{g}^*$ are *polarized* if there exists a vector $\eta \in \mathfrak{g}$ such that

$$\alpha_j(\eta) > 0 \quad \text{for} \quad j = 1,\dots,d. \tag{4.17}$$

Let us recall some standard facts from convex geometry that relate properties of the α_j's to properties of the map π, its image $C(\alpha_1, \dots, \alpha_d)$, and its level sets Δ_α.

PROPOSITION 4.14. *Let $\alpha_1, \dots, \alpha_d$ be elements of a vector space $\mathfrak{g}^*$. The following conditions are equivalent to each other:*

(1) *The α_j are polarized.*
(2) *The convex hull of the α_j's does not contain the origin $0 \in \mathfrak{g}^*$.*
(3) *The projection map (4.14) is proper.*
(4) *The set Δ_α is compact for every $\alpha \in \mathfrak{g}^*$.*
(5) *When $\alpha = 0$, the set Δ_α contains only the origin:*

$$\Delta_0 = \{0\}.$$

(6) *The set Δ_α is compact for some $\alpha \in C(\alpha_1, \dots, \alpha_d)$.*
(7) *The cone $C(\alpha_1, \dots, \alpha_d)$ is proper, that is, it does not contain a line.*

PROOF. Suppose that the elements α_j are polarized. Let $\eta \in \mathfrak{g}$ be a polarizing vector, so that $\alpha_j(\eta) > 0$ for all j. Then

$$m := \min\{\alpha_j(\eta)\}$$

is positive. If $\alpha = \pi(s) = \sum s_i \alpha_i$, then

$$\alpha(\eta) \geq m \sum s_i. \tag{4.18}$$

In particular, we cannot have $\alpha = 0$ when $\sum s_j = 1$. This proves (4.14). If $K \subset \mathfrak{g}^*$ is compact and $\pi(s) \in K$, then

$$M := \max_{\alpha \in K} \alpha(\eta) \geq m \sum s_j.$$

Hence, the preimage $\pi^{-1}(K)$ is contained in the simplex

$$\left\{ s \in \mathbb{R}^d_+ \mid \sum s_i \leq \frac{M}{m} \right\}.$$

Therefore, (4.14) implies (4.14) and (4.14).

Let us now assume that $D := \text{conv}\{\alpha_1, \dots, \alpha_d\}$ does not contain the origin. Then there exists a closed ball B around the origin which is still disjoint from D. By standard convexity theory, (see, e.g., [**Grü**, §2.2]), there exists a hyperplane which separates B from D, that is, there exists a vector η and a real number ϵ such that $\alpha(\eta) \geq \epsilon$ for all $\alpha \in D$ and $\alpha(\eta) \leq \epsilon$ for all $\alpha \in B$. Such ϵ must be positive. Hence, $\alpha(\eta) > 0$ for all $\alpha \in D$, and, in particular, for all $\alpha = \alpha_j$. This shows that the α_j's are polarized. Hence, (4.14) implies (4.14).

Clearly, if the projection map π is proper, then its level sets, Δ_α, are compact. Now, if $s = (s_1, \dots, s_d)$ is in the set

$$\Delta_0 = \left\{ s \mid \sum s_j \alpha_j = 0, \quad s_j \geq 0\ ,\ j = 1, \dots, d \right\}, \tag{4.19}$$

then so is $(ts_1, \dots, ts_d)$ for all $t \geq 0$. So Δ_0 is a cone in $\mathfrak{g}^*$. If this cone is also compact, it must be the zero cone. Condition (4.14) is proved. Condition (4.14) follows immediately.

We have established the implications (4.14) $\Leftrightarrow$ (4.14) $\Rightarrow$ (4.14) $\Rightarrow$ (4.14) $\Rightarrow$ (4.14) $\Rightarrow$ (4.14). Let us now show that the negation of (4.14) implies the negation of (4.14). Suppose that the convex hull of $\alpha_1, \dots, \alpha_d$ contains the origin. Let s_j be non-negative real numbers such that $\sum s_j = 1$ and $\sum s_j \alpha_j = 0$. For any

α, if $\sigma = (\sigma_1, \dots, \sigma_d)$ satisfies $\sum \sigma_j \alpha_j = \alpha$ and $\sigma_j \geq 0$ for all j, then so does $(\sigma_1 + ts_1, \dots, \sigma_d + ts_d)$ for all $t \geq 0$. In other words, the entire ray $\sigma + ts$, $t \geq 0$, is contained in Δ_α, and so Δ_α is not compact.

Having shown the equivalence of Conditions (4.14)–(4.14), it remains to show that these conditions are equivalent to Condition (4.14). If the cone $C(\alpha_1, \dots, \alpha_d)$ is not proper, then there exists $\beta \neq 0$ such that both β and $-\beta$ are in the cone. Let $s' = (s'_1, \dots, s'_d)$ and $s'' = (s''_1, \dots, s''_d)$ be non-negative coefficients such that $\beta = \sum s'_j \alpha_j$ and $-\beta = \sum s''_j \alpha_j$. Let $s = s' + s'' = (s_1, \dots, s_d)$. Then all s_j are non-negative and not all zero, and

$$\sum s_j \alpha_j = 0.$$

Dividing this equation by the sum $s_1 + \dots + s_d$, we conclude that the origin 0 is in the convex hull of $\alpha_1, \dots, \alpha_d$. Hence, the negation of (4.14) implies the negation of (4.14). Finally, assuming the negation of (4.14), let $s_1, \dots, s_d$ be non-negative real numbers such that $\sum s_j \alpha_j = 0$ and $\sum s_j = 1$. Up to permutation we may assume that $s_1 \neq 0$, so that

$$-\alpha_1 = \frac{1}{s_1} \sum_{j=2}^{d} s_j \alpha_j$$

is a non-negative combination of $\alpha_2, \dots, \alpha_d$. The cone $C(\alpha_1, \dots, \alpha_d)$ then contains both α_1 and $-\alpha_1$, so it contains the entire line through α_1. This proves the negation of (4.14). □

For our purposes, the most important property of moment maps is properness.

PROPOSITION 4.15. *The moment map*

$$\Phi(z) = \frac{1}{2} \sum_{j=1}^{d} |z_j|^2 \alpha_j$$

is η-polarized if and only if $\alpha_j(\eta) > 0$ for all j. The moment map Φ is proper if and only if it is polarized.

PROOF. The moment map is $\Phi = \pi \circ J$, where J and π are given by (4.13) and (4.14). Because J is proper and onto, $\Phi = \pi \circ J$ is proper if and only if π is proper and is η-polarized if and only if π is η-polarized. So we need to show that π is η-polarized if and only if the collection of α_j's is η-polarized and that π is proper if and only if it is polarized.

By Proposition 4.14, applied to $\mathbb{R}$ instead of $\mathfrak{g}^*$ and $\alpha_j(\eta)$ instead of α_j, the map π^η is proper if and only if all the $\alpha_j(\eta)$ have the same sign. In this case, clearly, π^η is bounded from below if and only if the $\alpha_j(\eta)$ are positive. This proves the first claim.

By Proposition 4.14, π is proper if and only if the α_j are η-polarized for some η, hence, if and only if π is η-polarized for some η. This proves the second claim. □

Recall that the Duistermaat–Heckman measure is the push-forward of the Liouville measure via the moment map. We will need the following formula for the Duistermaat–Heckman measure of a linear space.

PROPOSITION 4.16. *Suppose that the weights α_j are polarized and generate $\mathbb{Z}^*_G$. Then the density of the Duistermaat–Heckman measure on $\mathfrak{g}^*$ (with respect to the*

Lebesgue measure) is the function

$$\alpha \mapsto (2\pi)^d \operatorname{vol}(\Delta_\alpha),$$

where

$$\Delta_\alpha = \{s \in \mathbb{R}^d_+ \mid \sum s_j \alpha_j = \alpha\}, \tag{4.20}$$

Here, volumes are normalized as follows. The affine plane $A(\alpha)$ in which Δ_α lies contains a parallel shift of the lattice $\{m \in \mathbb{Z}^d \mid \sum m_j \alpha_j = 0\}$. The volume is normalized so that a fundamental chamber for this lattice has volume one. In $\mathfrak{g}^*$, Lebesgue measure is normalized so that the volume of $\mathfrak{g}^*/\mathbb{Z}^*_G$ is one.

Proof. The map $J(z_1, \dots, z_d) = \frac{1}{2}\left(|z_1|^2, \dots, |z_d|^2\right)$ pushes the Liouville measure on $\mathbb{C}^d$ to the Lebesgue measure on $\mathbb{R}^d_+$, multiplied by the constant coefficient $(2\pi)^d$. Indeed, the Liouville measure can be written as $|ds_1 \cdots ds_d d\theta_1 \cdots d\theta_d|$, where $s_j = \frac{1}{2} r_j^2$ and r_j and θ_j are polar coordinates. The map to $\mathbb{R}^d$ is the projection

$$(s_1, \dots, s_d, \theta_1, \dots, \theta_d) \mapsto (s_1 \dots, s_d),$$

and the Lebesgue measure on $\mathbb{R}^d_+$ is $ds_1 \cdots ds_d$. Hence, the Duistermaat–Heckman measure on $\mathfrak{g}^*$ is equal to the push-forward of the Lebesgue measure on $\mathbb{R}^d_+$ (up to $(2\pi)^d$) via the projection map (4.14). The proposition follows. □

We will also need the following fact concerning the further push-forward, to $\mathbb{R}$, by a single component of the moment map.

Proposition 4.17. *Suppose that the weights α_i are η-polarized. Then the measure on $\mathbb{R}$ obtained as the push-forward of the Liouville measure on $\mathbb{C}^d$ via $\Phi^\eta \colon \mathbb{C}^d \to \mathbb{R}$ has polynomial growth.*

Proof. Because the weights α_j are η-polarized, the number

$$m := \min\{\alpha_1(\eta), \dots, \alpha_d(\eta)\}$$

is positive. For all z,

$$\Phi^\eta(z) = \frac{1}{2}\sum_{j=1}^{d} |z_j|^2 \alpha_j(\eta) \geq \frac{1}{2} m \|z\|^2.$$

Thus, if $|\Phi^\eta(z)| \leq x$, then $\|z\|^2 \leq 2x/m$. Hence, the push-measure, evaluated on the interval $[-x, x]$, is no larger than the volume of the ball $\{z \in \mathbb{C}^d \mid \|z\|^2 \leq 2x/m\}$. Since this volume has polynomial growth in x, Proposition 4.17 follows. □

5. The right-hand side of the linearization theorems

The Hamiltonian linearization theorem involves certain data that is not given explicitly: the symplectic form and moment map on the right-hand side of (4.3). The theorem becomes more useful once we show that this data can be given by an explicit formula. In this section we give explicit formulas for two-forms and η-polarized moment maps on vector spaces and vector bundles. These can be used, by Lemma 4.6, as formulas for the right-hand side of the linearization theorem.

Our first formula applies to the case when a torus acts with isolated fixed points and the vector η is generic. In particular, we show that the forms $\omega_p^\#$ can be chosen symplectic.

PROPOSITION 4.18. *Consider the action of a torus G on $\mathbb{C}^d$ with weights $-\alpha_1, \dots, -\alpha_d \in \mathbb{Z}_G^*$:*

$$\exp \xi \colon (z_1, \dots, z_d) \mapsto \left(e^{-\sqrt{-1}\alpha_1(\xi)} z_1, \dots, e^{-\sqrt{-1}\alpha_d(\xi)} z_d \right). \tag{4.21}$$

Let $\eta \in \mathfrak{g}$ be a Lie algebra element such that $\alpha_j(\eta) < 0$ for $1 \le j \le r$ and $\alpha_j(\eta) > 0$ for $r + 1 \le j \le d$. Let $\epsilon_1 = \ldots = \epsilon_r = -1$ and $\epsilon_{r+1} = \ldots = \epsilon_d = 1$. Set $\alpha_j^{\#} = \epsilon_j \alpha_j$. Then

$$\omega^{\#} = \sum_{j=1}^{d} \epsilon_j dx_j \wedge dy_j$$

is a symplectic form with η-polarized moment map

$$\Phi^{\#}(z) = \Phi^{\#}(0) + \frac{1}{2} \sum_{j=1}^{d} |z_j|^2 \alpha_j^{\#}.$$

The vector η is called a *polarizing vector*; the weights $\alpha_j^{\#}$ are called the *polarized weights.*

PROOF. The proof is by a straight-forward computation. See Section 4. □

We recall that every linear torus action on a vector space V is isomorphic to an action of the form (4.21); see Section 4. Also, if the vector field η_V vanishes only at the origin (as is the case for $V = T_pM$ when $F = \{p\}$ on the right-hand side of (4.3)), then $\alpha_j(\eta) \neq 0$ for all j, and we may assume that $\alpha_j < 0$ exactly if $1 \le j \le r$. Therefore, any linear action that occurs on the right-hand side of (4.3) can be brought to the form as in Proposition 4.18.

We now consider the case when the set $M^\eta = \{\eta_M = 0\}$ is not discrete. (This happens if G acts with some non-isolated fixed points, or, more generally, M^G is discrete, but η is not generic). Suppose that the manifold M is equipped with an equivariant stable complex structure. Then NF becomes a complex vector bundle. In the rest of this section, we derive an explicit formula for the two-form $\omega^{\#}$ and η-polarized moment map $\Phi^{\#}$ on the total space of a G-equivariant complex vector bundle $\pi \colon E \to F$, where F is compact and connected, $E = NF$, and $\eta \in \mathfrak{g}$ is a Lie algebra element such that η_E vanishes exactly on the zero section: $E^\eta = F$. This provides an explicit formula for the right-hand side of the linearization theorem identity by Propositions 4.6 and D.19 (due to which an equivariant complex structure on NF carries the same information as a stable complex structure on F and a fiberwise complex structure on NF).

We construct $\omega^{\#}$ and $\Phi^{\#}$ using Sternberg's "minimal coupling" procedure, following [**GLS**]. Let $H \subseteq G$ be the closure of the one-parameter subgroup of G generated by η.

By decomposing the fiberwise action of H into isotypical components, we obtain a decomposition $E = E_1 \oplus \ldots \oplus E_s$ such that each E_j is a G-equivariant complex vector bundle over F and H acts on E_j fiberwise as multiplication by the inverse of a character $\rho_j \colon H \to S^1$. Let $\alpha_j = d\rho_j \in \mathfrak{h}^*$ be the corresponding weight. We may assume that $\alpha_j(\eta) < 0$ exactly if $1 \le j \le r$. Let $\epsilon_1 = \ldots = \epsilon_r = -1$ and $\epsilon_{r+1} = \ldots = \epsilon_s = 1$. Let $\alpha_j^{\#} = \epsilon_j \alpha_j$ be the "polarized weights".

Choose an invariant fiberwise Hermitian product on each E_j, and let $P_j \to F$ be the corresponding unitary frame bundle. Then P_j is a principal bundle with structure group $\mathrm{U}(m_j)$, where m_j is the rank of E_j, and $E_j = P_j \times_{\mathrm{U}(m_j)} \mathbb{C}^{m_j}$ is the associated bundle.

Let $P \to F$ be the fiberwise product $P = P_1 \times_F \ldots \times_F P_s$. This is a principal bundle for the group

$$K = \mathrm{U}(m_1) \times \ldots \times \mathrm{U}(m_s),$$

and E is the associated bundle $P \times_K \mathbb{C}^m$ with $m = m_1 + \ldots + m_s$.

We will obtain the two-form $\omega^{\#}$ by realizing E as a (pre-)symplectic reduction. Let Θ be a connection one-form on P; it associates to each $u \in TP$ an element $\Theta(u)$ of the Lie algebra $\mathfrak{k}$ of K. Pairing with the dual space $\mathfrak{k}^*$, we get a real valued one-form $\tilde{\Theta}$ on $P \times \mathfrak{k}^*$ given by $(u, \alpha) \mapsto \langle \Theta(u), \beta \rangle$ for $(u, \alpha) \in T_{(p,\beta)}(P \times \mathfrak{k}^*)$.

Consider the product $P \times \mathfrak{k}^* \times \mathbb{C}^m$ with the closed two-form

$$\tilde{\omega} = -d\tilde{\Theta} + \sum_{j=1}^{s} \sigma_j^{\#} + \pi^* \omega_F,$$

where σ_j is the standard symplectic form on the jth factor in $\mathbb{C}^m = \mathbb{C}^{m_1} \times \ldots \times \mathbb{C}^{m_s}$ and $\sigma_j^{\#} = \epsilon_j \sigma_j$. Let K act diagonally through the principal action on P. More precisely, we let $k \in K$ act by k^{-1} (to convert the principal right action into a left action), the coadjoint action on $\mathfrak{k}^*$, and the standard action on $\mathbb{C}^m$. The moment map for this action is $(p, \beta, z) \mapsto -\beta + \Phi_K(z)$, where $\Phi_K \colon \mathbb{C}^m \to \mathfrak{k}^*$ is the moment map for the linear K-action on $\mathbb{C}^m$ with symplectic form $\sigma_1^{\#} + \ldots + \sigma_s^{\#}$. The zero level set is $Z = \{(p, \beta, z) \mid \beta = \Phi_K(z)\} \cong P \times \mathbb{C}^m$ and the reduced space is $Z/K = P \times_K \mathbb{C}^m = E$. The two-form $\tilde{\omega}$ on Z descends to a two-form on the reduced space, whose pullback to $P \times \mathbb{C}^m$, which is obtained by substituting $\beta = \Phi_K(z)$ in the formula for $\tilde{\omega}$, is

$$\omega^{\#} = -d\langle \Theta, \Phi_K(z) \rangle + \sum_{j=1}^{s} \sigma_j^{\#} + \pi^* \omega_F.$$

The restriction of $\omega^{\#}$ to the zero section is ω_F.

The group G acts on P on the left by bundle automorphisms, with generating vector fields ξ_P, $\xi \in \mathfrak{g}$. This induces an action on $P \times_K \mathbb{C}^m = E$ whose moment map for the reduced two-form $\omega^{\#}$ is

$$\left(\Phi^{\#}\right)^{\xi}([p, z]) = \langle \Theta(\xi_P), \Phi_K(z) \rangle + \Phi_F^{\xi}(\pi(p)), \qquad \xi \in \mathfrak{g}.$$

The restriction of this moment map to the zero section is Φ_F.

The η-component of $\Phi^{\#}$ is

$$\begin{aligned} \left(\Phi^{\#}\right)^{\eta} &= \Phi_K^{\eta}(z) + \Phi_F^{\eta}(\pi(p)), \\ &= \tfrac{1}{2} \textstyle\sum_{j=1}^{s} \|z_j\|^2 \alpha_j^{\#}(\eta) + \Phi_F^{\eta}(\pi(p)), \end{aligned}$$

for $z = (z_1, \ldots, z_s)$ where $z_j \in \mathbb{C}^{m_j}$. Because $\alpha_j^{\#}(\eta) > 0$ for all j and F is compact, the η-component is proper and bounded from below. In other words, the moment map that we have constructed is η-polarized, as required.

6. The Duistermaat-Heckman and Guillemin-Lerman-Sternberg formulas

In Chapter 2 we introduced the Duistermaat–Heckman measure, which is the push-forward of the Liouville measure via the moment map, and its Fourier transform, the Duistermaat–Heckman oscillatory integral. There are explicit formulas which express these invariants in terms of infinitesimal data at the fixed points: these are the Duistermaat–Heckman formula for the oscillatory integral, and the Guillemin–Lerman–Sternberg formula for the push-forward measure. In this section we derive these formulas from the Hamiltonian linearization theorem.

The Duistermaat–Heckman measure is invariant under proper Hamiltonian cobordism by Theorem 2.24. Therefore, the Hamiltonian linearization theorem implies that the Duistermaat–Heckman measure is a sum of Duistermaat–Heckman measures associated to vector spaces, or to vector bundles. For instance, for a torus action with isolated fixed points, we get

$$\mathrm{DH}_{(M,\omega,\Phi)} = \sum_{p\in M^G} \mathrm{DH}_{(T_pM,\omega_p^{\#},\Phi_p^{\#})}\,. \tag{4.22}$$

We can derive explicit formulas for the summands in (4.22) from the explicit formulas for $(T_pM,\omega_p^{\#},\Phi_p^{\#})$, which were obtained in Section 5.

Namely, let $-\alpha_{1,p},\dots,-\alpha_{d,p}$ be the isotropy weights for the G-action on T_pM, and let $\alpha_{j,p}^{\#}$ be the corresponding polarized weights (see Proposition 4.18). Let ϵ_p be equal to 1 or -1 according to whether the number of j's such that $\alpha_{j,p}^{\#} = -\alpha_{j,p}$ is even or odd. At this point we note that the $\alpha_{j,p}$ are defined if M is stable complex (in particular, if M is symplectic), but otherwise each $\alpha_{j,p}$ is defined only up to a sign. However, ϵ_p is well defined if we require the isomorphism $T_pM\cong\mathbb{C}^d$ giving the form (4.21) to preserve the complex orientation. Moreover, the product $\prod_{j=1}^d \alpha_{j,p}$ is also well defined (as a function on $\mathfrak{g}$).

Let us now apply Proposition 4.16 to each T_pM and adjust for the orientations. Then (4.22) turns into the *Guillemin–Lerman–Sternberg formula*

$$\mathrm{DH}_{(M,\omega,\Phi)} \;=\; \text{Lebesgue measure} \;\times\; (2\pi)^d \sum_{p\in M^G} \epsilon_p \,\mathrm{vol}(\Delta_{p,\alpha}), \tag{4.23}$$

where

$$\Delta_{p,\alpha} = \left\{ s\in\mathbb{R}_+^d \mid \Phi(p) + \sum s_j\alpha_{j,p}^{\#} = \alpha \right\}.$$

This formula, when M is compact and ω is symplectic, was obtained in [**GLS**] by Guillemin, Lerman, and Sternberg. We allow M to be non-compact, as long as the moment map Φ is η-polarized.

Each summand of (4.23) is a measure on $\mathfrak{g}^*$ whose push-forward via $\eta\colon\mathfrak{g}^*\to\mathbb{R}$ has polynomial growth as was proved in Proposition 4.17. This implies that the function $e^{t\langle x,\eta\rangle}$, $x\in\mathfrak{g}^*$, can be integrated against this measure when $t<0$. If the number of fixed points is finite, integrating $e^{t\langle x,\eta\rangle}$ against the measures on the left- and right-hand sides of (4.23) gives the equality

$$\frac{1}{d!}\int_M e^{t\langle\Phi,\eta\rangle}\omega^d = \sum_{p\in M^G} \epsilon_p \left(-\frac{2\pi}{t}\right)^d \frac{e^{t\langle\Phi(p),\eta\rangle}}{\prod_{j=1}^d \alpha_{j,p}^{\#}(\eta)}. \tag{4.24}$$

Because $\prod_{j=1}^d \alpha_{j,p}^{\#} = \epsilon_p\prod_{j=1}^d \alpha_{j,p}$, the signs ϵ_p cancel, and (4.24) turns into

$$\frac{1}{d!}\int_M e^{t\langle\Phi,\eta\rangle}\omega^d = \sum_{p\in M^G} \left(-\frac{2\pi}{t}\right)^d \frac{e^{t\langle\Phi(p),\eta\rangle}}{\prod_{j=1}^d \alpha_{j,p}(\eta)}. \tag{4.25}$$

We proved the equality (4.25) for all $\eta\in\mathfrak{g}$ such that Φ is η-polarized and for all $t<0$. The left- and right-hand sides of (4.25) are well defined when η is replaced by any $\xi\in\mathfrak{g}\otimes\mathbb{C}$ which is not in the kernel of $\alpha_{j,p}$ for any j,p and where t is any nonzero complex number. Moreover, they are analytic functions of such ξ and t. Let us inspect more closely for which ξ and t the equation (4.25) holds. We return to the equality (4.23) of measures on $\mathfrak{g}^*$. For any ξ near η and t with negative real part, the function $e^{t\langle x,\xi\rangle}$ can still be integrated against each summand on the right-hand side of (4.23). Since there are finitely many summands, this function can also be integrated against the measure on the left-hand side of (4.23). This integration results in (4.25) with η replaced by ξ and now the formula holds for an open set of t's and ξ's. By analytic continuation, (4.25), with η replaced by ξ, holds for *all* $t \in \mathbb{C} \smallsetminus \{0\}$ and for *all* $\xi \in \mathfrak{g} \otimes \mathbb{C} \smallsetminus \bigcup_{j,p} \ker \alpha_{j,p}$. Setting $t = i$, we get the exact stationary phase formula of Duistermaat and Heckman:

$$\frac{1}{d!} \int_M e^{i\Phi} \omega^d = (2\pi i)^d \sum_{p \in M^G} \frac{e^{i\Phi(p)}}{\prod_{j=1}^d \alpha_{j,p}}, \tag{4.26}$$

as an equality between analytic functions on $\mathfrak{g} \smallsetminus \bigcup_{j,p} \ker \alpha_{j,p}$.

We recall the relevant conventions: the isotropy weights are $-\alpha_{j,p}$, and the moment map definition is $d\Phi^\xi = \iota(\xi_M)\omega$.

Duistermaat and Heckman originally stated and proved their formula (4.25) for M compact and under the additional assumption that the two-form ω is symplectic. With this assumption, the fixed points for the torus action are precisely the critical points for the moment map, and, if the action has isolated fixed points, the components Φ^ξ of the moment map, for generic ξ, are non-degenerate Morse functions. The *stationary phase approximation* then gives

$$\frac{1}{d!} \int_M e^{it\langle \Phi,\eta\rangle} \omega^d = \left(\frac{2\pi i}{t}\right)^d \sum_{p \in M^G} \frac{e^{it\langle \Phi(p),\eta\rangle}}{\prod \alpha_{j,p}(\eta)} + \begin{array}{c}\text{an error term}\\ \text{of order } t^{-d-1}\\ \text{as } t \to \infty.\end{array}$$

See [**GS1**, Chapter I]. The Duistermaat–Heckman formula (4.24) asserts that the error term is identically zero, so that the stationary phase approximation gives an *exact* expression for the oscillatory integral on the left.

Berline and Vergne, [**BV1**], showed that the Duistermaat–Heckman formula is a special case of the localization theorem for equivariant differential forms. Atiyah and Bott [**AB2**] interpreted the Duistermaat–Heckman formula in the context of topological equivariant cohomology. These arguments clarified that for compact manifolds the Duistermaat–Heckman formula is purely topological and holds without any non-degeneracy assumption on ω. (See Section 7 of Appendix C.)

Prato and Wu [**PW**] proved the Duistermaat–Heckman formula in the form (4.26) for *non-compact* symplectic manifolds whose fixed point set is finite and whose moment map has some component that is proper and bounded from below. In fact, their work served as one of the motivations for us to develop this theory. These results of Prato and Wu have been further extended by Paradan, [**Par1**], to the case where the fixed point set may have an infinite number of connected components.

Let us now consider the case where the fixed point set M^G is not discrete, or, more generally, where the polarizing vector $\eta \in \mathfrak{g}$ is not generic. Again, the Hamiltonian linearization theorem, together with the cobordism invariance of the Duistermaat–Heckman measure (Theorem 2.24), immediately imply a formula for

FIGURE 4.3. The Guillemin-Lerman-Sternberg formula for $\mathbb{CP}^2$

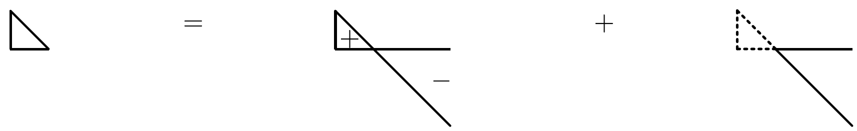

FIGURE 4.4. The Guillemin-Lerman-Sternberg formula for $\mathbb{CP}^2$, with a non-generic choice of "polarizing vector"

the Duistermaat–Heckman measure:

$$\mathrm{DH}_{(M,\omega,\Phi)} = \sum_{F \in \pi_0(M^\eta)} \mathrm{DH}_{(NF,\omega_F^\#,\Phi_F^\#)} \,. \tag{4.27}$$

The terms on the right-hand side were computed explicitly by Guillemin and Cannas da Silva in [**CG**]. Similarly to the case where the fixed points are isolated, integrating the measures on the left- and right-hand sides of (4.27) against the function $e^{t\langle \cdot,\eta\rangle}$ and taking analytic continuation, one obtains a version of the Duistermaat–Heckman formula (4.26) which applies when fixed points are not isolated or η is not generic.

A version of the Duistermaat–Heckman formula for *non-isolated* fixed points was already given in [**DH2**] and a version of the Guillemin–Lerman–Sternberg formula for non-isolated fixed points and for orbifolds was obtained by Guillemin and Cannas da Silva in [**CG**]. Our formulas generalize those to the cases where M is not compact or when M^η is larger than M^G. See Example 4.19.

EXAMPLE 4.19. Take $M = \mathbb{CP}^2$ with the Fubini–Study symplectic form, whose pullback to $S^5 \subset \mathbb{C}^3$ is the standard two-form $\sum_{j=1}^3 dx_j \wedge dy_j$. The linear action of $G = S^1 \times S^1$ on $\mathbb{C}^3$ by scalar multiplication on the first two coordinates induces an action on $\mathbb{CP}^2$. The image of the moment map is the Lebesgue measure on a triangle. The action on $\mathbb{CP}^2$ has three isolated fixed points; their images are the vertices of the triangle. Each of the summands DH_{T_pM} is the Lebesgue measure on the region between two rays and vanishes outside. The Guillemin-Lerman-Sternberg formula exhibits the triangle as a combination of three such "wedges"; see Figure 4.3. Notice that this involves a choice of direction in which the infinite rays are pointing; this is the choice of the "polarizing" vector η.

Let η be the generator of the first factor in $G = S^1 \times S^1$. The zero set M^η consists of one copy of $\mathbb{CP}^1$ and one isolated fixed point. The (generalized) Guillemin-Lerman-Sternberg formula gives the combination illustrated in Figure 4.4, in which the first summand is the Duistermaat–Heckman (signed) measure for the normal bundle of $\mathbb{CP}^1$ in $\mathbb{CP}^2$.