CHAPTER 1

Introduction

1. Topological aspects of Hamiltonian group actions

The objects we will be concerned with in this monograph, such as symplectic forms and moment maps, are easy to define and yet are of great complexity and depth. This complexity manifests itself in a variety of ways. For example the Darboux theorem is one of the most elementary theorems of symplectic geometry, but finding Darboux coordinates for a particular symplectic structure (for instance, a coadjoint orbit) can give rise to intricate problems in linear algebra and ordinary differential equations. To deal with this complexity one is often forced to narrow one's focus and concentrate on specific aspects of these objects, for example, invariance properties of moment maps (energy or momentum conservation laws) or topological data (the cohomology class of the form or the homotopy class of the associated almost complex structure).

In this monograph, our focus will be on global properties of Hamiltonian torus actions and their connection with topology.

1.1. Global invariants of Hamiltonian group actions. Over the course of the last twenty years it has gradually become clear that this topic, global properties of Hamiltonian group actions, has more to do with topology, and less to do with symplectic geometry, than was previously realized.

The first inkling of this came from the Duistermaat–Heckman theorem, [**DH1**]. This theorem states that the oscillatory integral for the moment map of a torus action on a symplectic manifold is exactly equal to the leading term of its asymptotic expansion. Hence, all terms of higher order in the expansion vanish. This provides a formula for the Fourier transform of the push-forward of the Liouville measure by the moment map in terms of the fixed points of the action. More explicitly consider an action of the circle, G, on a compact symplectic manifold (M^{2n}, ω) with isolated fixed points and moment map $\Phi\colon M \to \mathbb{R}$. Then the Duistermaat–Heckman formula reads

$$\frac{1}{n!}\left(\frac{1}{2\pi}\right)^n \int_M e^{\Phi}\omega^n = \sum_{p\in M^G} \frac{e^{\Phi(p)}}{\prod \alpha_{j,p}}.$$

Here M^G is the fixed point set and $\alpha_{1,p}, \dots, \alpha_{n,p}$ are the weights of the linearized action of G on T_pM at a fixed point p.

Although the Duistermaat–Heckman theorem was originally proved in the symplectic setting, it was soon discovered by Berline and Vergne and Atiyah and Bott, [**BV1, AB1**], that the theorem is just a particular case of a general localization formula in equivariant cohomology for torus actions. This formula, of a purely topological nature, equates the integral of an equivariant cohomology class over M and the sum of the integrals of this class over the components of the fixed point set, with corrections coming from the actions on the normal bundles to the components. Explicitly,

$$\int_M u = \sum_F \int_F \frac{u|_F}{e(\mathcal{V}_F)}.$$

Here u is an equivariant cohomology class on M, the summation runs over the components F of the fixed point set, and $e(\mathcal{V}_F)$ is the equivariant Euler class of the normal bundle to F. When applied to the class $u = e^{-(\omega-\Phi)}$, the localization theorem turns into the Duistermaat–Heckman formula.

A second example of this "topologizing" of the global theory of Hamiltonian torus actions involves geometric quantization and indices of Dirac operators associated with symplectic structures. The Atiyah–Singer index theorem expresses the index of such an operator as the integral of a certain characteristic class. (See, e.g., [**ASe, ASi**] and [**BGV, Du, Gil**].) When the operator is invariant under a compact group action, this index becomes a virtual representation and its character can be evaluated as the integral of an equivariant cohomology class. The geometric quantization $\mathcal{Q}(M,\omega)$ is defined as the virtual representation, i.e., index, of the Dirac operator. In this case, the index theorem takes the form

$$\mathcal{Q}(M,\omega) = \int_M e^{\omega/2\pi}\,\mathrm{Td}\,(M),$$

where $\mathrm{Td}\,(M)$ is the Todd class of M. In the equivariant situation $\mathrm{Td}\,(M)$ should be replaced by the equivariant Todd class and ω by the equivariant symplectic form $\omega-\Phi$, where Φ is the moment map. Note that in this formula the integrand depends only on the cohomology class of the symplectic structure and the Chern classes of the almost complex structure associated with the symplectic form. This Chern class is actually just an invariant of the "stable complex structure" associated with this almost complex structure.[1] Moreover, a cohomology class, a stable complex structure, and an orientation are sufficient to define a Dirac operator with the "correct" index. A variant of this is "Spin^c quantization", which only depends on the cohomology class and the orientation.

When the group acting on the symplectic manifold is abelian, the index is determined by the fixed point data. An explicit formula, the Atiyah–Bott–Lefschetz fixed point theorem, [**AB1**], can be obtained by applying the localization theorem to the integrand above.

A third example is the *quantization commutes with reduction* theorem. This asserts that the G-invariant part of the quantization of a symplectic G-manifold is equal to the quantization of the reduction at the zero level of the moment map. Various versions of this theorem, often referred to as [Q,R]=0, have been proved in the last decade. (See Section 10 of Chapter 8 for a detailed survey and references.) This result is also essentially of a topological nature although the situation is now more subtle. As we have seen, geometric quantization only requires a cohomology class and an equivariant stable complex structure. However, symplectic reduction requires more. Namely, we need a moment map. We get that from an equivariantly closed two-form. (Recall that an equivariant two-form is a pair consisting of an invariant two-form and an equivariant map to $\mathfrak{g}^*$. This form is equivariantly closed if and only if the two-form is closed and the map satisfies Hamilton's equation. The

[1] Since we will need this class in situations when M is not symplectic or not even–dimensional, we recall that a stable complex structure is just a complex structure on the sum $TM \oplus \mathbb{R}^k$ for some k.

form is *not* assumed to be non-degenerate, but we will still refer to the map as a moment map.) Under certain additional assumptions, purely topological proofs of the quantization commutes with reduction theorem are obtained in [**Met3, Par2**] using equivariant K-theory. In this book we give an alternative topological proof, in the case where G is a torus and the fixed points are isolated.

A fourth example is the non-abelian localization theorem of Jeffrey and Kirwan, [**JK1**]. This result, which is closely related to the quantization commutes with reduction theorem, expresses the integral over the reduced space of an equivariant cohomology class coming from a Hamiltonian G-manifold in terms of the fixed points of the action.[2] For G a torus, a topological interpretation of the Jeffrey–Kirwan theorem can be found, for example, in [**GGK1**].

1.2. Geometry of Hamiltonian group actions as a branch of equivariant topology. To summarize, the four theorems which we described above are theorems about objects which one encounters in symplectic geometry but are basically theorems in equivariant topology. Moreover, it would not be an exaggeration to say that the area of symplectic geometry which deals with global properties of Hamiltonian group actions is, to a large extent, a branch of equivariant topology. This topology, however, involves manifolds that are equipped with somewhat unconventional structures. The choice of structures depends on the kind of question one is asking, but some common traits are already clear. For the Duistermaat–Heckman formula, it is sufficient to consider manifolds equipped with an equivariant cohomology class of degree two. In questions of geometric quantization, one needs an additional structure, such as an equivariant stable complex or Spin^c structure. In the "quantization commutes with seduction" theorem, symplectic reduction is involved, and for this one needs a moment map. Thus one has to replace the cohomology class by a closed equivariant two-form. In other words, the structure considered now is a triple (M, ω, Φ) consisting of an oriented G-manifold (not necessarily of even dimension), a G-invariant closed two-form ω (not necessarily of maximal rank), and an equivariant moment map Φ, such that ω and Φ are related by the Hamilton equations (which means that the formal difference $\omega - \Phi$ is a closed equivariant two-form). We will henceforth refer to such a triple as a *Hamiltonian G-manifold*.

These are not the only possible choices of structures, and, perhaps, not the optimal ones. Below, in Section 4, we propose yet one more refinement, which allows one to separate the moment map from the cohomology class.

Note that the cohomological data and the stable complex data reflect the two different roles that a symplectic form plays: a closed form determines a cohomology class, and a non-degenerate form determines an almost complex (and hence, stable complex) structure. As a result of separating the two roles, a considerable amount of information is lost, but the setting becomes simpler.

There is one more implication of non-degeneracy which we have entirely ignored so far. Namely, the non-degeneracy of a symplectic form implies a non-degeneracy condition on the moment map. This condition can also be looked at from a topological point of view, and we will do so in Section 4.2. Non-degeneracy is often essential in the local study of Hamiltonian actions, e.g., in singular reduction, [**ACG, BL, SL**]. The local questions are closely connected with (and

[2] In spite of its name, the Jeffrey–Kirwan non-abelian localization theorem is *not* an extension of the Berline–Vergne–Atiyah–Bott abelian localization to non-abelian groups.

to some extent motivated by) the investigation of stability of relative equilibria of mechanical systems. Here, the reduced energy-momentum and Lagrangian block diagonalization methods, [**Lew, MSLP, SLM**], are among the most efficient and versatile. This is apparently due to the fact that these methods use only moment maps and Hamiltonians rather than symplectic structures.

We conclude this section by pointing out that we are not attempting to make the indefensible claim that *all* of the global theory of Hamiltonian torus actions can be reduced to questions in topology. There are many genuinely symplectic results, in which the symplectic form itself, not just the topological structures one can define from it, plays a fundamental role. For example, classifying Hamiltonian actions and determining whether they admit a Kähler structure are symplecto-geometric, not topological, questions. See e.g., [**Ka4, KT2, To, Ww**]. Finally, there are many topics that we will not touch on in this book: topics in symplectic topology and Poisson geometry which address problems of a different nature from those that we will be considering below and which require methods different from ours.

2. Hamiltonian cobordism

The Duistermaat–Heckman formula and the index formula involve integration over the underlying manifold. By Stokes' formula, the Duistermaat–Heckman integral and the index are both cobordism invariant, provided that the structures necessary for producing these invariants extend over the cobording manifold. The same applies to many other global invariants of Hamiltonian group actions which are obtained by integrating characteristic classes over the manifold or the reduced manifold.

This has led us to consider "cobordism theories" of oriented manifolds equipped with G-actions and equivariant cohomology classes of degree two (or closed equivariant two-forms) and, if necessary, G-equivariant stable complex structures,[3] see [**GGK1**].

To be specific, let (M_r, ω_r, Φ_r), for $r = 0, 1$, be oriented $2n$-dimensional compact Hamiltonian G-manifolds.[4] These manifolds are said to be *cobordant* if there exists a compact oriented $2n+1$-dimensional Hamiltonian G-manifold with boundary, (W, ω, Φ), such that

(i) $\partial W = -M_0 \sqcup M_1$, and
(ii) $i_r^*\omega = \omega_r$ and $i_r^*\Phi = \Phi_r$,

where $i_r \colon M_r \to W$ is the inclusion map. We call such a cobordism a *Hamiltonian cobordism*. In this definition, only the equivariant cohomology classes $[\omega_r - \Phi_r]$ matter: the same cobordism theory would result from considering manifolds equipped with equivariant cohomology classes rather than two-forms. If, in addition, each M_r is equipped with a G-equivariant stable complex structure, we assume that these structures extend to W.

[3] A word of warning on terminology: what is traditionally called "symplectic cobordism" in algebraic topology is cobordisms between pairs (X_r, J_r), for $r = 0, 1$, where X_r is a compact oriented manifold and J_r is a stable reduction of the structure group of the tangent bundle of X_r to the complex symplectic group. This is related, but only remotely so, to the type of cobordism we are considering here. This cobordism is also different from symplectic cobordism between contact manifolds, which is a partial order rather than an equivalence relation, studied in symplectic topology.

[4] If ω_r is symplectic, the orientation of M_r need not coincide with the orientation defined by ω_r^n.

Now it is clear that the Duistermaat–Heckman integral and the equivariant index are invariants of the cobordism. The same is true for any other invariant obtained by integrating the product of equivariant Chern classes with the equivariant cohomology class of the two-form. We refer to these invariants as *mixed characteristic numbers*. These are the invariants we are primarily concerned with in this book. Note that mixed characteristic numbers are not really numbers but elements of $H^*(BG)$, i.e., polynomials in $\dim G$ variables.

This cobordism theory was studied in [**GGK1**] and applied to the problems that we discussed in Section 1. Additionally, we proved in [**GGK1**] that "cobordism commutes with reduction", i.e., that if Hamiltonian G-manifolds are cobordant then so are the reduced spaces, with the induced structures. (Since a regular reduced space is only an orbifold, the cobordism involved here is an orbifold cobordism, cf. [**Dr**].)

Having defined Hamiltonian cobordisms we face the problem of calculating the Hamiltonian cobordism ring. This problem is tangential to the main purpose of this book and we defer its detailed discussion to Appendix H.

We note that the results of [**GGK1**] were obtained simultaneously with the development of Shaun Martin's cobordism technique [**Mart1**] to study the cohomology ring of reduced spaces.

3. The linearization theorem and non-compact cobordisms

3.1. The linearization theorem. In this section we will focus on the case where G is a torus. The *fixed point data* for a G-manifold M are the fixed point set M^G and the linearized G-action on the normal bundle $\mathcal{V}_{M^G}$ to M^G. If M is equipped with an equivariant stable complex structure, M^G inherits this structure and the normal bundle is a genuine complex bundle with the G-action preserving the complex structure. In addition, when M is a Hamiltonian G-manifold, the restriction of the equivariant cohomology class to M^G is also assumed to be a part of the fixed point data.

The Berline–Vergne–Atiyah–Bott localization theorem ensures that every equivariant characteristic number of a (stable complex, Hamiltonian) G-manifold can be expressed in terms of the fixed point data. These characteristic numbers are also invariants of cobordism and, conjecturally, determine the cobordism class. Thus one may expect that the cobordism class of M is determined by the fixed point data. This is indeed true, as was shown by Gusein-Zade in the early 70s, [**GZ1**]. This fact can be expressed more succinctly as asserting that M is cobordant to its fixed point data:

THEOREM 1.1 (Linearization Theorem, I). *Let G be a torus, let M be a compact (stable complex) Hamiltonian G-manifold, and let F_i, $i = 1, \dots, k$, be the connected components of the fixed point set M^G. Then M is equivariantly cobordant to the disjoint union of the normal bundles to F_i:*

$$M \sim \bigsqcup \mathcal{V}_{F_i}. \tag{1.1}$$

In particular, if M^G is finite, M is equivariantly cobordant to the disjoint union of the tangent spaces T_pM, for $p \in M^G$.

A word of warning: For this statement to be meaningful a somewhat unorthodox definition of the symbol "$\sim$" is required. Indeed, the right-hand side of (1.1) is non-compact and the introduction of non-compact objects into a cobordism theory usually has a trivializing effect. For example, every compact manifold M is cobordant to the empty set via the non-compact cobordism $M \times [0,1)$. This difficulty is fundamental and not easy to fix. For instance, one might think that one can get around this difficulty by equipping manifolds and cobordisms with proper functions. However, this does not solve the problem because a pair (M, f) consisting of a compact manifold and a proper function is cobordant to the empty set via the cobordism $(M \times [0,1), f + 1/(1-t)^2)$, where $t \in [0,1)$.[5] Thus we have to impose additional constraints to make non-compact cobordisms non-trivial and thus make sense of the right-hand side of (1.1). The rest of this section is devoted to doing this.

The first version of Theorem 1.1 was proved in [**GGK1**]. In that version we used the moment map on M and symplectic cutting, [**Ler1**], to compactify the spaces $\mathcal{V}_{F_i}$ or, to be more accurate, find compact approximations of these spaces. When the fixed points are isolated, this version of the theorem asserts that M is cobordant to a union of certain twisted projective spaces associated with the fixed point data. Here by a twisted projective space we mean the quotient of the sphere by an action of S^1 with arbitrary non-zero weights. (Note also that a similar theorem is obtained in [**Mart1**].)

Thus in [**GGK1**], (1.1) is not yet a literal identity. The cobordism is still a cobordism of Hamiltonian stable complex *compact* G-orbifolds. Notice also that if $\widetilde{\mathbb{CP}}^n$ is one of the twisted projective spaces occurring in this cobordism, it comes equipped with a stable complex structure which can be incompatible with its natural Kähler structure (even if M is symplectic). This shows that stable complex structures in our theory are a matter not of convenience but of necessity.

The first version of Theorem 1.1 in which (1.1) becomes a literal cobordism appeared in [**Ka3**] and involved abstract moment maps. This notion we will describe in Section 4. First, however, we outline a different way to make rigorous sense of (1.1).

3.2. Non-compact Hamiltonian cobordism.

Two Hamiltonian G-manifolds with proper moment maps are *properly cobordant* if there exists a Hamiltonian cobordism between them such that the moment map on the cobording manifold is proper.

Clearly, we have a map from the set of compact cobordism classes to the set of proper cobordism classes. For manifolds with stable complex structures, this map is one-to-one; see [**GGK2**].[6] Hence by passing from compact to proper cobordisms we lose no information about compact cobordism classes.

Now we are in a position to state a rigorous version of Theorem 1.1. For the sake of simplicity we restrict our attention to the case of isolated fixed points.

Theorem 1.2 (Linearization Theorem, II). *Let G be a torus and let (M, ω, Φ) be a compact (stable complex) Hamiltonian G-manifold with isolated fixed points. Then there exist equivariant closed forms $\omega_p - \Phi_p$ on T_pM for $p \in M^G$ with Φ_p*

[5] Although in such a theory all compact manifolds are cobordant to zero, the theory does detect the ends of a non-compact manifold. Thus the theory is not completely trivial.

[6] Strictly speaking the result of [**GGK2**] concerns stable complex cobordisms with abstract moment maps. However, it is not hard to modify the proof to obtain the Hamiltonian version of this result.

proper (and equivariant stable complex structures) such that (M, ω, Φ) *is properly cobordant to the disjoint union of* (T_pM, ω_p, Φ_p).

The forms ω_p can always be chosen symplectic. However, even when ω is symplectic, the stable complex structure on T_pM induced by ω may fail to be compatible with ω_p. We prove several versions of the linearization theorem in Chapter 4.

Finally, we note that cobordism techniques discussed here are used in [**MW2, MW3**] to study Hamiltonian loop group actions and moduli spaces of flat connections.

3.3. Reduction and cobordism. The "cobordism commutes with reduction" principle asserts that reductions of cobordant manifolds are cobordant. Combined with the linearization theorem this implies that the reduced space M_{red} at a regular value of the moment map is cobordant in the class of orbifolds to a union of toric orbifolds, provided that the fixed points are isolated:

$$M_{\mathrm{red}} \sim \bigsqcup_{p\in M^G} (T_pM)_{\mathrm{red}}.$$

We emphasize here that these toric orbifolds $(T_pM)_{\mathrm{red}}$ may carry non-standard stable complex structures even when M is symplectic.

4. Abstract moment maps and non-degeneracy

4.1. Abstract moment maps. An abstract moment map is an equivariant mapping $M \to \mathfrak{g}^*$ which has some of the formal properties of genuine moment maps but without a two-form. More explicitly, let M be a G-manifold and Φ a G-equivariant mapping of M into $\mathfrak{g}^*$. For a closed subgroup H of G, let $\mathfrak{h}$ be the Lie algebra of H and $\Phi^H\colon M \to \mathfrak{h}^*$ the composition of Φ and the natural projection $\mathfrak{g}^* \to \mathfrak{h}^*$. Note that if Φ is a genuine moment map, Φ^H is the moment map for the H-action on M. Hence, Φ^H is constant on the connected components of M^H. This property of genuine moment maps is taken as the definition of an abstract moment map.

Definition 1.3. An equivariant map $\Phi\colon M \to \mathfrak{g}^*$ is an *abstract moment map* if for every closed subgroup H of G, the map Φ^H is constant on connected components of M^H.

We emphasize that this definition does not require M to be equipped with a two-form. However, as we showed in [**GGK3**], if G is a torus and Φ an abstract moment map, there always exists, at least locally, an equivariant closed two-form whose moment component is Φ. We also gave a necessary and sufficient condition for the existence of a global such two-form.

As we have pointed out above, abstract moment maps were introduced in [**Ka3**] to make the statement of the linearization theorem precise. Namely, consider oriented G-manifolds equipped with proper abstract moment maps and, if necessary, G-equivariant stable complex structures and/or equivariant cohomology classes of degree two. A proper cobordism between two such manifolds is defined in the obvious way: the moment map on the cobording manifold is required to be proper. We refer to this cobordism as a *cobordism of proper abstract moment maps.* The linearization Theorem 1.2 holds with proper cobordisms of Hamiltonian G-manifolds replaced by cobordisms of this type.

The main advantage of working with abstract moment maps rather than with Hamiltonian G-manifolds is that the moment map is no longer attached to the equivariant cohomology class. One can still carry out reduction at a regular value of an abstract moment map. Moreover, an equivariant cohomology class induces an ordinary cohomology class on the reduced space, and an equivariant stable complex structure descends to a stable complex structure on the reduced space. Thus, the "cobordism commutes with reduction" principle still holds for proper cobordism of abstract moment maps.

Examples of abstract moment maps have come up in other contexts in symplectic geometry not related to cobordisms or global properties of Hamiltonian group actions. For instance, the locked momentum map introduced by Lewis *et al*, [**Lew, SLM**], in geometric mechanics in the early nineties is an abstract moment map in our sense, i.e., it is not associated with a symplectic form; see Example 3.11. Abstract moment maps are implicitly used in [**LT2**] in intersection cohomology calculations.

4.2. Non-degenerate abstract moment maps. A moment map associated with a genuine symplectic form satisfies an additional non-degeneracy condition which comes from the non-degeneracy of the symplectic form. We say that an abstract moment map $\Phi\colon M \to \mathfrak{g}^*$ is *non-degenerate* if for every $\xi \in \mathfrak{g}$ the ξ-component Φ^ξ is a Morse–Bott function and the critical set of Φ^ξ is exactly the zero set of the vector field ξ_M induced on M by ξ. It turns out that when G is a torus this condition is necessary and sufficient for Φ to be locally associated with a symplectic form; see Appendix G.

The well-known convexity theorem asserts that when G is a torus and M is compact and symplectic, the moment map image $\Phi(M)$ is a convex polytope and the level sets of Φ are connected, [**At3, GS2**]. The proof of this theorem is purely Morse–theoretic and relies on the fact that moment maps for symplectic forms are non-degenerate. Hence the convexity theorem also holds for non-degenerate abstract moment maps. Likewise, the Kirwan theorem (which asserts that a symplectic Hamiltonian G-manifold is equivariantly formal, i.e., $H_G^*(M) = H^*(M) \otimes H^*(BG)$) is also true for manifolds with non-degenerate abstract moment maps since formality is a consequence of the Morse–theoretic properties described above. This circle of questions is discussed in detail in Appendix G.

Furthermore, it is reasonable to expect that some of the singular reduction theorems, [**BL, SL**], will have counterparts for manifolds with non-degenerate abstract moment maps. For example, it appears to be true that $\Phi^{-1}(a)/G$ is a stratified space for all $a \in \mathfrak{g}^*$.

We conclude this section by pointing out that in the non-abelian case the theory of abstract moment maps is entirely undeveloped (see, however, [**Brad**]) and it is not clear at all which of the results discussed above can be generalized to actions of non-abelian groups.

5. The quantum linearization theorem and its applications

5.1. The quantum linearization theorem. The linearization theorem says that a compact Hamiltonian manifold M with isolated fixed points is cobordant to the disjoint union of the tangent spaces T_pM at the fixed points p of the action.

The quantum analogue of the linearization theorem simply asserts that the geometric quantization $\mathcal{Q}(M)$ is equal, as a virtual representation, to the sum of the quantizations of the tangent spaces:

$$\mathcal{Q}(M) = \sum_{p \in M^G} \mathcal{Q}(T_p M).$$

Here, as in the linearization theorem, M is equipped with an equivariant two-form and an equivariant stable complex structure. The tangent spaces T_pM are also equipped with such structures, inherited from M as in Theorem 1.2.

Since the linear spaces T_pM are not compact, extra care should be exercised when their quantizations are defined. Here we adopt the definition of $\mathcal{Q}(T_pM)$ using $\mathcal{L}^2$-Dolbeault cohomology as in, for example, [**Man1, Man2**]. With this definition, every irreducible representation of G occurs in $\mathcal{Q}(T_pM)$ with finite multiplicity. The quantum linearization theorem asserts that the sum of these multiplicities over all $p \in M^G$ is equal to the multiplicity with which the representation occurs in $\mathcal{Q}(M)$.

As stated, the quantum linearization theorem is a simple consequence of the Atiyah–Bott Lefschetz theorem. A more general version of the quantum linearization theorem is proved by Braverman, [**Brav4**]. Braverman's theorem does not require M to be compact: the theorem applies, for example, to a stable complex G-manifold with finite fixed point set and proper abstract moment map.

5.2. Quantization commutes with reduction. As an application of cobordism techniques we will give in Chapter 8 a detailed proof of the "quantization commutes with reduction" theorem in the case where G is a torus and the fixed points are isolated.

We prove two versions of this theorem. One version is purely topological. To state it, consider a stable complex compact Hamiltonian G-manifold M and let α be an integral regular value of the moment map $\Phi\colon M \to \mathfrak{g}^*$. Then the theorem asserts that

$$\mathcal{Q}(M)^\alpha = \mathcal{Q}(M_\alpha).$$

Here $\mathcal{Q}(M)^\alpha$ is the multiplicity with which α occurs in $\mathcal{Q}(M)$ and M_α is the reduction of M at α. The second version of the theorem is symplectic. In this version we do not require α to be regular but M is then required to be symplectic; the quantization of M_α is defined using a suitable desingularization.

The proofs of both of these theorems follow the same path. The idea is to reduce the problem to the case of a linear space by using the cobordism invariance of quantization and the linearization theorem. For the sake of simplicity, we suppose that the fixed points of the action are isolated. To begin with, let us also assume that α is a regular value for the moment maps on all the T_pMs. Then the linearization theorem yields the cobordism

$$M_\alpha \sim \bigsqcup_{p \in M^G} (T_p M)_\alpha.$$

Hence, since the quantization is an invariant of cobordism,

$$\mathcal{Q}(M_\alpha) = \sum_{p \in M^G} \mathcal{Q}((T_p M)_\alpha).$$

On the other hand, the quantum linearization theorem asserts that

$$\mathcal{Q}(M) = \sum_{p \in M^G} \mathcal{Q}(T_pM).$$

Thus it suffices to show that quantization commutes with reduction for the linear spaces T_pM:

$$\mathcal{Q}(T_pM)^\alpha = \mathcal{Q}((T_pM)_\alpha).$$

The quantization of a linear space is well-understood and an explicit expression for $\mathcal{Q}(T_pM)^\alpha$ is given in terms of a partition function. The reduced space $(T_pM)_\alpha$ is a toric orbifold equipped with its standard symplectic structure and a stable complex structure. These structures are not necessarily compatible with each other, and one of the main complications in the proof will be dealing with this incompatibility.

We show that $\mathcal{Q}((T_pM)_\alpha)$ is equal to the quantization of $(T_pM)_\alpha$ with respect to the *standard* complex structure and a new two-form ω obtained from the standard symplectic form by a certain shift. Hence,

$$\mathcal{Q}((T_pM)_\alpha) = \sum (-1)^k \dim H^{0,k}((T_pM)_\alpha; \mathbb{L}),$$

where $\mathbb{L}$ is a holomorphic line bundle over $(T_pM)_\alpha$ whose first Chern class is $[\omega]$. (The cohomology spaces on the right-hand side are computed in [**Od**] for an arbitrary line bundle $\mathbb{L}$.)

The first term $H^{0,0}((T_pM)_\alpha; \mathbb{L})$ is just the space of global holomorphic sections of $\mathbb{L}$. These can be identified with those holomorphic functions on a certain open and dense subset $W \subset T_pM$ which transform correctly under the torus action. We calculate $\dim H^{0,0}((T_pM)_\alpha; \mathbb{L})$ combinatorially, in terms of partition functions, and conclude that this dimension is exactly $\mathcal{Q}(T_pM)^\alpha$.

The next step is to show that $\dim H^{0,k}((T_pM)_\alpha; \mathbb{L}) = 0$ if $k > 0$; this is one of the complications we alluded to above. (Notice that this result is akin to the Kodaira vanishing theorem.) We argue combinatorially again. Roughly speaking, the cohomology in question is identified with the relative simplicial cohomology of a pair (Δ, L), where Δ is the moment polytope and L is a contractible subset of its boundary, and by elementary topology this cohomology vanishes.

When α is regular for M but not necessarily for the tangent spaces T_pM, we perturb the moment map so that the above construction goes through. (The "right" perturbation is obtained by shifting Φ by a small abstract moment map rather than by a constant.)

A more general version of the "quantization commutes with reduction" theorem for non-compact manifolds is obtained by Braverman in [**Brav4**].

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