CHAPTER 2

Hamiltonian cobordism

1. Hamiltonian group actions

1.1. Definitions. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. A smooth G action on a manifold M gives rise to a Lie algebra anti-homomorphism,

$$\mathfrak{g} \to \mathrm{Vect}(M),$$

associating to a Lie algebra element $\xi \in \mathfrak{g}$ the vector field ξ_M that generates the action on M of the one-parameter subgroup $\{\exp(t\xi) \mid t \in \mathbb{R}\}$ of G.

Let ω be a closed G-invariant two-form on M. A *moment map* is a smooth map

$$\Phi\colon M \to \mathfrak{g}^*$$

that is equivariant with respect to the G-action on M and the coadjoint action on $\mathfrak{g}^*$ and such that the components $\Phi^\xi = \langle \Phi, \xi \rangle$ satisfy Hamilton's equation,

$$d\Phi^\xi = \iota(\xi_M)\omega, \tag{2.1}$$

for all $\xi \in \mathfrak{g}$. In other words, for every tangent vector η we have $\eta\Phi^\xi = \omega(\xi_M, \eta)$, where we let η act as a derivation on the function Φ^ξ.

When G is the circle group S^1, we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ with $\mathbb{R}$, and the moment map equation becomes $d\Phi = \iota(\xi_M)\omega$ for $\xi = \frac{\partial}{\partial\theta}$. (See Appendix A for our conventions.)

Example 2.1. Consider the plane $\mathbb{R}^2$ with the standard symplectic form $dx \wedge dy = r dr \wedge d\theta$. Let the circle group S^1 act by rotations, with generating vector field $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial\theta}$. Then $-\frac{1}{2}(x^2+y^2) = -\frac{1}{2}r^2$ is a moment map. More generally, on $\mathbb{C}^n = (\mathbb{R}^2)^n$ with $\omega = \sum dx_i \wedge dy_i$, the S^1-action with weights $m_1, \dots, m_n \in \mathbb{Z}$ has the generating vector field $\sum m_j \frac{\partial}{\partial\theta_j}$ and moment map $-\frac{1}{2}\sum m_j(x_j^2+y_j^2)$.

A G-action on (M,ω) is called *Hamiltonian* if it admits a moment map Φ. The moment map is then determined by (2.1) uniquely up to translation. A *Hamiltonian*

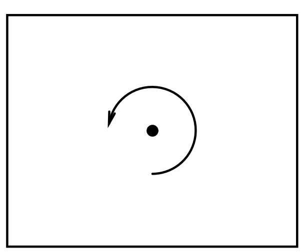

Figure 2.1. Rotations of the plane

G-manifold is a triple (M, ω, Φ), where M is an oriented manifold with a G action, ω is a closed two-form, and $\Phi\colon M \to \mathfrak{g}^*$ is a moment map.

EXAMPLE 2.2. Consider the torus $S^1 \times S^1 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ with the symplectic form induced from $\mathbb{R}^2$. Let the circle group S^1 act by rotations of the first factor. This action is not Hamiltonian: in the periodic coordinates $(x \mod 2\pi)$ and $(y \mod 2\pi)$, the moment map equation is $d\Phi = \iota(\frac{\partial}{\partial x})dx \wedge dy = dy$, which has no global periodic solution.

1.2. Hamiltonian mechanics. The two-form ω is often assumed to be *symplectic*, i.e., not only closed but also non-degenerate. If ω is non-degenerate, the vector fields ξ_M (and hence the action, if G is connected) are determined by the moment map through (2.1).

Non-degeneracy implies that the dimension of M is even, and is equivalent to the condition that the top wedge product ω^n, where $n = \frac{1}{2}\dim M$, never vanishes. This induces an orientation on M. The *Liouville measure* is defined by the integration of the volume form $\omega^n/n!$ with respect to the symplectic orientation.

For example, the standard symplectic form

$$\omega_{\text{std}} = \sum_{i=1}^{n} dx_i \wedge dy_i \tag{2.2}$$

on $\mathbb{R}^{2n}$ is non-degenerate, and $\omega_{\text{std}}^n = n! dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$. Locally, *any* symplectic form can be brought to the form (2.2) by an appropriate choice of local coordinates by Darboux's theorem. (See [**McDSa**, Section 3.2].) Writing $x_i = q_i$ and $y_i = p_i$, the moment map equation (2.1) becomes Hamilton's equations from mechanics: the vector field ξ_M generates the flow that solves the differential equations

$$\dot{q}_i = \partial\Phi^\xi/\partial p_i \quad \text{and} \quad \dot{p}_i = -\partial\Phi^\xi/\partial q_i. \tag{2.3}$$

When Φ^ξ is replaced by the classical Hamiltonian

$$H \;=\; \frac{1}{2m}p^2 + U(q) \;=\; \text{kinetic } + \text{ potential energy},$$

the equations below become the time flow in phase space:

$$\dot{q} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial U}{\partial q}.$$

More generally, the phase space of a classical mechanical system is modeled by a symplectic manifold (M, ω), and the dynamics of the system is the $\mathbb{R}$-action on M whose moment map is the Hamiltonian (total energy) function $H \in C^\infty(M)$. The *kinematics* of the system, namely, the "shape" of phase space, often involve symmetries, which mathematically are given by a proper group action (see Appendix B). The "conjugate momenta" for these symmetries are no other than the moment map coordinates for this action. *Noether's theorem* asserts that these quantities are conserved by the time flow if the Hamiltonian is preserved by the symmetry. In this formulation, the proof is immediate: if η_M is the vector field that generates the time flow, so that $\omega(\eta_M) = dH(\cdot)$, then

$$\frac{d\Phi^\xi}{dt} = \eta_M \Phi^\xi = \omega(\xi_M, \eta_M) = -\xi_M H = 0.$$

EXAMPLE 2.3. For N particles in $\mathbb{R}^3$ of masses $m_1, \ldots, m_N$, the velocity phase space is $T(\mathbb{R}^{3N}) = \mathbb{R}^{6N}$, with position coordinates x^i, y^i, z^i, and velocity coordinates $\dot{x}^i$, $\dot{y}^i$, $\dot{z}^i$, and with a symplectic form

$$\omega = \sum_{i=1}^{N} m_i (dx^i \wedge d\dot{x}^i + dy^i \wedge d\dot{y}^i + dz^i \wedge d\dot{z}^i).$$

The group $\mathbb{R}^3$ acts by translations with generating vector fields $\sum \frac{\partial}{\partial x^i}$, $\sum \frac{\partial}{\partial y^i}$, and $\sum \frac{\partial}{\partial z^i}$, and moment map

$$\Phi(x_1, \ldots, \dot{z}_N) = \sum_{i=1}^{N} m_i (\dot{x}^i, \dot{y}^i, \dot{z}^i),$$

which associates to each state of the system its total linear momentum. The group SO(3) acts by rotations with generating vector fields ξ_1, ξ_2, ξ_3, where $\xi_1 = \sum_{i=1}^{N} y^i \partial/\partial z^i - z^i \partial/\partial y^i + \dot{y}^i \partial/\partial \dot{z}^i - \dot{z}^i \partial/\partial \dot{y}^i$, etc., and moment map

$$\Phi = \sum_{i=1}^{N} m_i \left(y^i \dot{z}^i - z^i \dot{y}^i \ , \ z^i \dot{x}^i - x^i \dot{z}^i \ , \ x^i \dot{y}^i - y^i \dot{x}^i \right),$$

which associates to each state of the system its total angular momentum.

1.3. The Duistermaat–Heckman measure. In this book we do *not* assume that ω is symplectic. When we write a Hamiltonian G-manifold as a triple (M, ω, Φ), we assume that the orientation and G-action on M are also given. Even if ω happens to be non-degenerate, we do not insist that it be compatible with the orientation on M. We define the Liouville measure, as before, by integrating the form $\omega^n/n!$ with respect to the given orientation; this is now a *signed* measure.

DEFINITION 2.4. The *Duistermaat–Heckman measure* on $\mathfrak{g}^*$ is the push-forward of Liouville measure by the moment map. We denote it DH_M.

REMARK 2.5. It is convenient to work with the differential form (of mixed degree)

$$\exp \omega = 1 + \omega + \frac{1}{2!} \omega \wedge \omega + \frac{1}{3!} \omega \wedge \omega \wedge \omega + \ldots .$$

With the convention that $\int_M \beta = 0$ if the degree of β is different than the dimension of M, Liouville measure is given by integration of $\exp \omega$.

EXAMPLE 2.6. Take the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with its standard area form. Let the circle group act by rotations, fixing the north and south poles. The moment map is the height function $\Phi(x, y, z) = z$. The Duistermaat–Heckman measure is 2π times Lebesgue measure on the interval $[-1, 1]$.

This amounts to Archimedes' observation, that the area of the strip $\{(x, y, z) \in S^2 \mid a \leq z \leq b\}$ is $2\pi(b - a)$, assuming $-1 \leq a < b \leq 1$. If we fix the height difference $b - a$, this can be a short but wide strip near a pole of S^2, or it can be a thin but long strip near the equator. Equivalently, the standard area form on S^2 is given in cylindrical coordinates by $\omega = d\theta \wedge dz$.

In some contexts, one can forget the two-form and just keep track of the action and the moment map, or, more specifically, the moment map values at the fixed points. For instance, if M is compact and G is a torus acting with isolated fixed

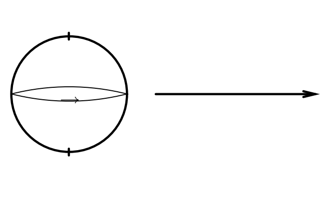

FIGURE 2.2. Rotations of S^2 and moment map

points, the moment map values at the fixed points determine global invariants such as the Duistermaat–Heckman measure (see Chapter 4) and the geometric quantization (see Chapter 6). The following important example illustrates the role of fixed points:

EXAMPLE 2.7. Let ω be a rotation-invariant two-form on S^2. The Duistermaat–Heckman measure is 2π times Lebesgue measure on the interval $[\Phi(q), \Phi(p)]$, where p and q are the north and south poles and Φ is a moment map. In particular, the total area is $2\pi(\Phi(p) - \Phi(q))$. We leave the proof to the reader as an exercise.

1.4. Existence of moment maps. We turn to the question of whether a moment map exists.

EXAMPLE 2.8 (Exact moment maps). Let $\omega = -d\mu$, where μ is a G-invariant one-form. Then the function $\Phi \colon M \to \mathfrak{g}^*$ defined by $\Phi^\xi = \mu(\xi_M)$ is a moment map.

PROOF. Because μ is invariant, its Lie derivative along ξ_M vanishes, $L_{\xi_M}\mu = 0$, for all $\xi \in \mathfrak{g}$. By Cartan's homotopy formula for the Lie derivative, $L_{\xi_M} = d\iota(\xi_M) + \iota(\xi_M)d$, this implies the moment map equation

$$d\left(\iota(\xi_M)\mu\right) = -\iota(\xi_M)d\mu.$$

□

In particular, if ω is exact and invariant and G is compact, a moment map always exists. (The reason is that there always exists μ as in Example 2.8; see Corollary B.13.) What happens if we remove the exactness assumption?

Just the fact that the two-form ω is closed and invariant implies that the one-forms $\iota(\xi_M)\omega$ are closed. Indeed, by Cartan's formula,

$$d\iota(\xi_M)\omega = \left(d\iota(\xi_M) + \iota(\xi_M)d\right)\omega = L_{\xi_M}\omega = 0.$$

If the one-forms $\iota(\xi_M)\omega$ are *exact*, one can choose functions Φ^ξ that satisfy the moment map equation (2.1) for basis elements ξ of $\mathfrak{g}$ and take their linear combinations. Therefore, if we ignore the equivariance requirement, a moment map always exists locally, and the obstruction for it to exist globally lies in the first de Rham cohomology of M.

We will be particularly interested in torus actions. For these, the equivariance condition amounts to the moment map being invariant (because the coadjoint action is trivial), and this condition automatically follows from Hamilton's equation (2.1):

PROPOSITION 2.9. *Suppose that G is a torus and $\Phi \colon M \to \mathfrak{g}^*$ satisfies $d\Phi^\xi = \iota(\xi_M)\omega$ for all $\xi \in \mathfrak{g}$. Then Φ is G-invariant.*

PROOF. The tangent space at p to the orbit $G \cdot p$ is $\{\xi_M|_p \mid \xi \in \mathfrak{g}\}$. Hence, it is enough to show that $L_{\xi_M}\Phi^\eta = 0$ for all $\xi, \eta \in \mathfrak{g}$. Let us first show that $L_{\xi_M}\Phi^\eta$ is constant along an orbit. For this we need to show that $L_{\zeta_M}L_{\xi_M}\Phi^\eta = 0$ for all $\zeta \in \mathfrak{g}$. Indeed,

$$\begin{aligned} L_{\zeta_M}L_{\xi_M}\Phi^\eta &= L_{\zeta_M}\iota(\xi_M)d\Phi^\eta = L_{\zeta_M}\iota(\xi_M)\iota(\eta_M)\omega = L_{\zeta_M}\omega(\eta_M,\xi_M) \\ &= (L_{\zeta_M}\omega)(\eta_M,\xi_M) + \omega([\zeta_M,\eta_M],\xi_M) + \omega(\eta_M,[\zeta_M,\xi_M]), \end{aligned}$$

where the last equality is Leibniz's rule. All three summands are zero.

This shows that $L_{\xi_M}\Phi^\eta$ is constant on each orbit. By compactness, each orbit contains a point where Φ^η is maximal. At this point, the derivative of Φ^η at any direction along the orbit must be zero. In particular, $L_{\xi_M}\Phi^\eta = 0$. □

In general, the obstruction to the existence of a moment map lies in the Lie algebra cohomology. This implies that a moment map always exists if G is semisimple, even when $H^1(M) \neq 0$. Concretely, the obstruction to the existence of a map Φ satisfying Hamilton's equation (2.1) is the element of $H^1(\mathfrak{g}) \otimes H^1(M) = H^1(\mathfrak{g}; H^1(M))$ represented by the cocycle $c\colon \xi \to [\iota(\xi_M)\omega]$. (The fact that this is a cocycle, that is, that $c([\xi,\eta]) = 0$, follows from the formula $\iota([\xi_M,\zeta_M]) = L_{\xi_M}\iota(\zeta_M) - \iota(\zeta_M)L_{\xi_M}$.) If this obstruction is zero, the obstruction to the existence of an *equivariant* moment map is the element of $H^2(\mathfrak{g},\mathbb{R})$ represented by the cocycle $\Phi^{[\xi,\eta]} - \{\Phi^\xi, \Phi^\eta\}$ where Φ is any map satisfying (2.1) and $\{,\}$ is the Poisson bracket (see Section 1.5). See, e.g., [**CW**, Section 7.2], [**Can**, Section 2.6], and [**Gin4**] for the treatment of this question in the more general context of actions on Poisson manifolds,

1.5. The Poisson algebra. If ω is non-degenerate, it associates to each smooth function $f \in C^\infty(M)$ a vector field $\xi_f \in \mathrm{Vect}(M)$ such that $df(\cdot) = \omega(\xi_f, \cdot)$. The *Poisson bracket*

$$\{f,g\} := \xi_g f$$

defines a Lie algebra structure on $C^\infty(M)$, such that the map $f \mapsto \xi_f$ is an anti-Lie homomorphism from $C^\infty(M)$ to $\mathrm{Vect}(M)$. If G is connected, the equivariance condition on a moment map $\Phi\colon M \to \mathfrak{g}^*$ is equivalent to the map

$$\mathfrak{g} \to C^\infty(M), \qquad \xi \mapsto \Phi^\xi \tag{2.4}$$

being a Lie algebra homomorphism.

Let us briefly recall the proofs of these facts. We have $\omega(\xi_f, \xi_g) = \xi_g f = \{f,g\}$. This implies that the Poisson bracket is anti-symmetric. We have

$$\xi_f\omega(\xi_g,\xi_h) = \xi_f\{g,h\} = \{\{g,h\},f\} \tag{2.5}$$

and

$$\begin{aligned} \omega([\xi_f,\xi_g],\xi_h) = -[\xi_f,\xi_g]h &= -\xi_f\xi_g h + \xi_g\xi_f h \\ &= -\{\{h,g\},f\} + \{\{h,f\},g\}. \end{aligned} \tag{2.6}$$

For any three vector fields a, b, and c, we have

$$\begin{aligned} d\omega(a,b,c) &= a\omega(b,c) + \text{ cyclic permutations} \\ &\quad - \omega([a,b],c) - \text{ cyclic permutations}. \end{aligned} \tag{2.7}$$

By (2.5), (2.6) and (2.7),

$$
\begin{aligned}
d\omega(\xi_f, \xi_g, \xi_h) &= \{\{g,h\},f\} + \text{ cyclic permutations} \\
&\quad + \{\{h,g\},f\} + \text{ cyclic permutations} \\
&\quad - \{\{h,f\},g\} - \text{ cyclic permutations} \\
&= C - C + C,
\end{aligned} \tag{2.8}
$$

where

$$C := \{\{f,g\},h\} + \text{ cyclic permutations}.$$

Since ω is closed, (2.8) implies $C = 0$, so the Poisson bracket satisfies the Jacobi identity. Also, as in (2.6),

$$[\xi_f, \xi_g]h + \xi_{\{f,g\}}h = (\{\{h,g\},f\} - \{\{h,f\}\},g) + \{h,\{f,g\}\} = -C = 0,$$

showing that $f \mapsto \xi_f$ is an anti-Lie homomorphism. Finally, because $\{\Phi^\xi, \Phi^\eta\} = -\xi_M \Phi^\eta$, the map (2.4) is a Lie algebra homomorphism if and only if $\xi_M \Phi^\eta = \Phi^{-[\xi,\eta]}$, which exactly means that $\Phi\colon M \to \mathfrak{g}^*$ is equivariant with respect to the action of the Lie algebra of G (see Section 1.6 of Appendix B).

1.6. Poisson algebras for degenerate two-forms. The notion of the Poisson algebra of a symplectic manifold generalizes to manifolds (M,ω) equipped with closed two-forms.

The motivation for this generalization comes from *quantization.* Given an (integral) symplectic manifold, the Kirillov–Kostant "pre-quantization recipe" produces a "(pre-)quantization" of the entire Poisson algebra $C^\infty(M)$. Given an (integral) Hamiltonian G-action, this recipe produces a "(pre-)quantization" of the moment map components Φ^ξ, even if the two-form is degenerate. See Chapter 6 for details. An attempt to unite these two notions of "quantization" naturally leads to an infinite-dimensional algebra which is canonically associated to any manifold M and closed two-form ω. We denote this algebra $\mathcal{P}(M,\omega)$.

As a vector space,

$$\mathcal{P}(M,\omega) = \{(f,v) \in C^\infty(M) \times \mathrm{Vect}(M) \mid df = \iota_v \omega\}. \tag{2.9}$$

Multiplication is defined by

$$(f,v)\cdot(g,u) = (fg, fu + gv). \tag{2.10}$$

A Lie bracket is defined by

$$[(f,v),(g,u)] = (L_u f, -[u,v])\,. \tag{2.11}$$

Note that $L_u f = -L_v g = \omega(v,u)$. With these structures, $\mathcal{P}(M,\omega)$ is a Poisson algebra, in the sense that it is simultaneously a commutative algebra and a Lie algebra and that the Lie bracket is a derivation with respect to the multiplication operation. We leave the details as an exercise to the reader.

Example 2.10. When ω is symplectic,

$$\mathcal{P}(M,\omega) \cong C^\infty(M) \quad \text{via} \quad (f,v) \mapsto f.$$

When $\omega = 0$,

$$\mathcal{P}(M,0) = \mathbb{R} \times \mathrm{Vect}(M).$$

For any ω, there is a short exact sequence

$$0 \to \mathbb{R} \to \mathcal{P}(M,\omega) \to \mathrm{Ham}(M) \to 0,$$

where

$$\mathrm{Ham}(M) = \{v \in \mathrm{Vect}(M) \mid \iota_v\omega \text{ is exact }\}.$$

As in the symplectic case, any $(f, v) \in \mathcal{P}(M, \omega)$ generates a flow on M (through v), and we have the energy conservation law

$$L_v f = 0$$

and Poincaré's integral invariant

$$L_v \omega = 0.$$

In fact, the canonical homomorphism

$$\mathcal{P}(M, \omega) \to \mathrm{Vect}(M), \qquad (f, v) \mapsto v$$

gives rise to an infinitesimal $\mathcal{P}(M, \omega)$-action on (M, ω).

In Section 4 of Chapter 6 we will identify $\mathcal{P}(M, \omega)$ as the algebra of infinitesimal symmetries of a "pre-quantization data" (P, Θ) for (M, ω), assuming $\frac{1}{2\pi}[\omega]$ is integral.

Example 2.11. For any Hamiltonian G-manifold (M, ω, Φ), we have a homomorphism of Lie algebras

$$\mathfrak{g} \to \mathcal{P}(M, \omega), \qquad \xi \mapsto (\Phi^\xi, \xi_M).$$

Remark 2.12. This construction generalizes to Dirac structures. A Dirac structure is a sub-bundle $\mathcal{V}$ of $T^*M \otimes TM$ that satisfies a certain integrability condition. Two forms, Poisson structures, and foliations are special cases of Dirac structures. For a Dirac structure, the Hamilton equation in the definition (2.9) has to be replaced by the condition that (df, v) be a section of $\mathcal{V}$. We leave the details as an exercise to the reader. (See, [**Co**] for the definition of Dirac structures and [**Wei3**] for further references.)

2. Hamiltonian geometry

Geometry of moment maps for torus actions on symplectic manifolds is rich in beautiful results, some of which we will survey here. Throughout this section, G is a torus acting on a compact symplectic manifold (M, ω) with moment map $\Phi\colon M \to \mathfrak{g}^*$.

We begin with the *convexity theorem* of Atiyah, Guillemin, and Sternberg:

Theorem. *The image $\Phi(M)$ is a convex polytope in the vector space $\mathfrak{g}^*$.*

We call $\Phi(M)$ the *moment polytope*.

More explicitly, the fixed point set has finitely many connected components, and the moment map takes a constant value on each such component. The moment map image is the convex hull of these values:

$$\Phi(M) = \mathrm{conv}\{\Phi(p) \mid p \in M^G\}. \tag{2.12}$$

There is a companion theorem of the convexity theorem, involving the set $\Phi(M)_{\mathrm{reg}}$ of regular values of Φ in $\Phi(M)$. We call its connected components the *alcoves* of the moment polytope.

Theorem. *The alcoves are open convex polytopes.*

A closely related result is the following *connectedness theorem*:

Theorem. *The level sets $\Phi^{-1}(\alpha)$, $\alpha \in \mathfrak{g}^*$, are connected.*

Another related result is the *stability theorem*:

THEOREM. *As a map to* $\Phi(M)$*, the moment map is open.*

Equivalently, as p varies within a moment fiber $\Phi^{-1}(\alpha)$, the moment map image of a neighborhood of p remains the same and equal to a small neighborhood of $\alpha \in \mathfrak{g}^*$ in $\Phi(M)$. This is the reason for the name "stability". We emphasize that this theorem is not local: it asserts that the moment map can have "corners" and "foldings" only on the boundary of $\Phi(M)$. For example, if $G = S^1$, the end-points of $\Phi(M)$ are the only local maxima and minima of Φ.

The convexity and connectedness theorems are due to Atiyah and Guillemin and Sternberg. The stability theorem for compact manifolds is proved in [**Sj3**, Theorem 6.5]. For proper moment maps to open convex sets, see [**LMTW**] or [**CDM**]. We will discuss the convexity theorem, as well as other consequences of non-degeneracy, in Appendix G.

From the *local normal form* [**GS7, Marl**], which completely describes a Hamiltonian space near an orbit, one deduces local versions of the above theorems: a neighborhood of an orbit maps to an open subset of a convex polytope; the level sets are locally connected; the moment map image of a neighborhood of p remains the same as p varies continuously within $\Phi^{-1}(\alpha)$.

Then, one argues one way or another that the result remains true globally. For instance, one may employ Morse theory, as in [**GS2**], using the following result:

THEOREM. *If* $G = S^1$*, the moment map* $\Phi\colon M \to \mathbb{R}$ *is a perfect Morse-Bott function. If the fixed points are isolated,* Φ *is a Morse function.*

This result, which follows from the local normal form (cf. [**GS6**, §32]), essentially goes back to Frankel [**F**]. For G a torus, each component of Φ is Morse-Bott. Note that the critical points of Φ are exactly the fixed points for the action, as follows from (2.1). Alternatively, to prove the convexity and related theorems, one may argue by induction on the dimension of the torus, as in [**At3**], or one may look for shortest paths in M with respect to the pull-back of a metric on $\mathfrak{g}^*$, as in [**CDM**].

We note that "patching together" the local results is similar to the following elementary but non-trivial theorem:

> Let A be a subset of $\mathbb{R}^n$ which is closed, connected, and locally convex (every point has a neighborhood U such that $A \cap U$ is convex). Then A is convex.

We state an important special case of the normal form theorem:

THEOREM. *Any fixed point* $p \in M$ *has a neighborhood which is isomorphic to a neighborhood of the origin in* $\mathbb{C}^n$*, with the standard symplectic form* $\sum dx_i \wedge dy_i$*, and with the torus acting through an inclusion into* $(S^1)^n$*.*

This theorem is a version of the equivariant Darboux theorem (see, e.g., [**GS6**]) and can be proved by the standard Moser's homotopy argument.

The moment map on $\mathbb{C}^n$ is $\Phi(p) + \frac{1}{2}\sum |z_i|^2 \alpha_{i,p}$ where $-\alpha_{i,p} \in \mathbb{Z}_G^*$ are the *isotropy weights* or just *weights* at p (see Appendix A). Its image is the *moment cone*,

$$C_p = \Phi(p) + \sum \mathbb{R}_+ \alpha_{i,p}.$$

Now suppose that M is compact. The stability theorem and the local normal form theorem imply that there exists a neighborhood U of $\Phi(p)$ such that

$$\Phi(M) \cap U = C_p \cap U. \tag{2.13}$$

The moment polytope is equal to the intersection of the moment cones:

$$\Phi(M) = \bigcap_{p \in M^G} C_p. \tag{2.14}$$

This follows from (2.13), together with the facts that $\Phi(M)$ is a convex polytope and that its vertices are a subset of $\{\Phi(p) \mid p \in M^G\}$.

Equations (2.12) and (2.14) are important characterizations of the moment polytope $\Phi(M)$ which play a role in the proof of the "quantization commutes with reduction" theorem; see Chapter 8. When ω is not symplectic, the moment image $\Phi(M)$ is not a polytope and appears to be of little interest. However, in the presence of a stable complex structure (see Appendix D), the isotropy weights, and hence the moment cones, are well defined. The intersection $\cap C_p$ and the convex hull $\operatorname{conv}\Phi(M^G)$ might now be different from each other, and they both provide meaningful replacements for the moment polytope.

The Morse theoretic properties of the moment map imply that, for a torus action with isolated fixed points, the cohomology class $[\omega]$ is determined by the moment map values at the fixed points. To see this, restrict to a sub-circle action with the same fixed points. Because its moment map is a perfect Morse function, the second homology is generated by invariant two-spheres descending from fixed points. The value of $[\omega]$ on such a two-sphere is determined by the moment map values at the fixed points as in Example 2.7.

Often, the moment map encodes much more than the cohomology class of the two-form. For instance, the moment map plays a central role in classification theorems.

THEOREM (Delzant, [**De**]). *If $\dim G = \frac{1}{2}\dim M$, the moment polytope $\Phi(M)$ determines the Hamiltonian manifold (M, ω, Φ) up to isomorphism.*

Here, isomorphism means equivariant symplectomorphism that preserves the moment map. For a torus G, compact Hamiltonian G-manifolds with $\dim G = \frac{1}{2}\dim M$ are called *Delzant spaces*. The moment polytope $\Delta = \Phi(M)$ is *regular*, meaning that the edge vectors at each vertex can be normalized to be a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}_G^*$ in $\mathfrak{g}^*$. An explicit construction of a Delzant space (M, ω, Φ) from a given regular polytope Δ is described in Section 5 of Chapter 5. In this Delzant situation, the level sets of Φ are precisely the G-orbits; moreover, the moment map gives a *diffeomorphism* $M/G \to \Delta$, meaning that a function on Δ extends to a smooth function on $\mathfrak{g}^*$ exactly if it pulls back to a smooth invariant function on M, as can be seen from the local normal form. The set of regular values of Φ consists of only one alcove: the interior of Δ. The Duistermaat–Heckman measure is equal to Lebesgue measure on Δ times $(2\pi)^n$ (cf. Example 2.6). Hence, the Liouville volume of a Delzant space is proportional to the Euclidean volume of its moment polytope:

$$\int_M \exp \omega = (2\pi)^n \mathrm{vol}_{\mathrm{Euc}} \Delta.$$

Delzant's theorem was generalized to orbifolds by Lerman and Tolman [**LT1**]: a Hamiltonian G-orbifold is determined by the moment polytope Δ, together with

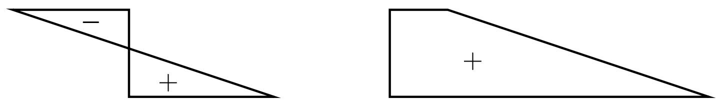

FIGURE 2.3. Twisted polytope for a Delzant space

an integer for each facet F of Δ, which encodes the orbifold singularity along the codimension 2 submanifold $\Phi^{-1}F \subset M$.

The *complexity* of a Hamiltonian G-manifold is $\frac{1}{2}\dim M - \dim G$, which measures how far we are from the Delzant situation. For actions of complexity one, classification results can be found in [**Ka4, KT2, KT3**]. The complexity is equal to half the dimension of a non-empty regular reduced space $\Phi^{-1}(\alpha)/G$. It also effects the Duistermaat–Heckman measure, whose density function is given on each alcove by a polynomial of degree at most the complexity; see Section 3 of Chapter 5.

Delzant's theorem has an interesting generalization in the case that ω is degenerate. Over the set $\mathfrak{g}^*_{\mathrm{reg}}$ of regular values of Φ, the quotient $\overline{M} = \Phi^{-1}\mathfrak{g}^*_{\mathrm{reg}}/G$ is a manifold, and the map $\overline{\Phi}\colon \overline{M} \to \mathfrak{g}^*_{\mathrm{reg}}$ is a proper local diffeomorphism. The *degree function* associates to each regular value $\alpha \in \mathfrak{g}^*_{\mathrm{reg}}$ the number of points in the preimage of α, counted with appropriate signs. If M is a toric variety, M/G is homeomorphic to a polytope with boundary homeomorphic to S^{n-1}, the moment map restricts to a map $\overline{\Phi}\colon S^{n-1} \to \mathfrak{g}^* \smallsetminus \{\alpha\} \cong \mathbb{R}^n \smallsetminus \{\text{point}\}$, and the winding number of this map is equal to the degree at α. The Duistermaat–Heckman measure is equal to Lebesgue measure times the degree function times $(2\pi)^n$. For example, in Figure 2.3, the polygon on the right corresponds to some Delzant space, and the "twisted polygon" on the left corresponds to a non-symplectic two-form on the same space. The Duistermaat–Heckman measure is equal to Lebesgue measure on the bottom triangle and minus Lebesgue measure on the top triangle, times $(2\pi)^2$. The same degree function also gives the multiplicities of weights for the quantization of M. See [**KT1, GK, Masu, Hat3**].

3. Compact Hamiltonian cobordisms

Let G be a connected Lie group and (M, ω, Φ) a compact Hamiltonian G-manifold. The *Duistermaat–Heckman integral* is $\int_M \exp(i\Phi + \omega)$, that is, the function $I\colon \mathfrak{g} \to \mathbb{C}$ given by

$$I(\xi) = \int_M e^{i\Phi^\xi} \omega^n/n!, \tag{2.15}$$

where $n = \frac{1}{2}\dim M$. Passing to the variable $\alpha = \Phi(p)$,

$$I(-\xi) = \int_{\alpha \in \mathfrak{g}^*} e^{-i\langle \alpha, \xi \rangle}\, \mathrm{DH}_M = \widehat{\mathrm{DH}}_M(\xi)$$

is the Fourier transform of the Duistermaat–Heckman measure (see Definition 2.4).

By the method of stationary phase, the integral of $e^{it\varphi}$ is the sum over the critical points, $\{p \mid d\varphi|_p = 0\}$, of certain expressions that depend on the Hessian $d^2\varphi|_p$, plus a remainder term that decays as $t \to \infty$ at least as fast as t^{-n-1}. If ω is symplectic, the critical points of $\varphi = \Phi^\xi$ are exactly the fixed points for the G-action, as follows from (2.1). The astounding discovery of Duistermaat and Heckman is that in this case the remainder term is identically zero.

In Section 6 of Chapter 4 we give the resulting *Duistermaat–Heckman exact stationary phase formula* (which involves summation over fixed points and is valid for all invariant closed two-forms). We deduce it from the "Hamiltonian linearization theorem". In Section 7 of Appendix C we deduce the Duistermaat–Heckman formula from the Atiyah-Bott-Berline-Vergne localization formula in equivariant cohomology.

Perhaps the first indication that cobordisms may be relevant for Hamiltonian geometry was the realization by Guillemin and Sternberg, in the mid-eighties [**GS5**], that the Duistermaat–Heckman integral is an invariant of cobordism:

LEMMA 2.13. *Let (M_0, ω_0, Φ_0) and (M_1, ω_1, Φ_1) be Hamiltonian G-manifolds that are cobordant in the following sense: there exists a compact Hamiltonian G-manifold with boundary (W, ω, Φ) such that*

$$\partial W = -M_0 \sqcup M_1,$$

$$\omega|_{\partial W} = \omega_0 \sqcup \omega_1,$$

and

$$\Phi|_{\partial W} = \Phi_0 \sqcup \Phi_1.$$

Then their Duistermaat–Heckman integrals are equal.

The minus sign in $-M_0$ indicates reversal of orientation.

PROOF. The equality

$$\int_{M_0} e^{i\Phi_0^\xi} \omega_0^n / n! = \int_{M_1} e^{i\Phi_1^\xi} \omega_1^n / n! \tag{2.16}$$

follows from Stokes' theorem, noting that the integrands extend to a closed $2n$-form, $e^{i\Phi^\xi}\omega^n/n!$, on W. The closedness of this $2n$-form is a special case of the following lemma. □

LEMMA 2.14. *Let (W, ω, Φ) be a $(2n+1)$-dimensional Hamiltonian G-manifold. Then for any smooth function $\varphi \colon \mathfrak{g}^* \to \mathbb{R}$, the $2n$-form $(\varphi \circ \Phi)\omega^n$ on W is closed.*

PROOF. By the chain rule, the total derivative of this form is $\langle d\varphi \circ \Phi, d\Phi \wedge \omega^n \rangle$, where $\langle , \rangle$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Note that $d\varphi \circ \Phi$ is a $\mathfrak{g}$-valued function and $d\Phi \wedge \omega^n$ is a $\mathfrak{g}^*$-valued $(2n+1)$-form. For any ξ,

$$d\Phi^\xi \wedge \omega^n = \frac{1}{n+1} \iota(\xi_M) \omega^{n+1} = 0,$$

because ω^{n+1} is a $(2n+2)$-form on a $(2n+1)$-manifold. □

Lemma 2.13 was proved by Guillemin and Sternberg [**GS5**] in the context of of equivariant cohomology. We recall a relevant definition:

DEFINITION 2.15. An equivariant differential two-form on a G-manifold M is, by definition, a formal sum $\omega + \Phi$, where ω is an invariant two-form on M and Φ is a smooth equivariant function from M to $\mathfrak{g}^*$. This equivariant form is *equivariantly closed* if $(d + \iota(\xi_M))(\omega + \Phi^\xi) = 0$ for all $\xi \in \mathfrak{g}$. The equivariant form $\omega + \Phi$ is *equivariantly exact* if there exists an invariant one-form μ such that $\omega + \Phi^\xi = (d + \iota(\xi_M))\mu$ for all $\xi \in \mathfrak{g}$. The second equivariant cohomology, denoted $H^2_G(M)$, is the quotient of the space of equivariantly closed two-forms by the subspace of equivariantly exact two-forms. See Appendix C.

EXAMPLE 2.16. $\omega - \Phi$ is equivariantly closed if and only if Φ is a moment map for ω; it is equivariantly exact if and only if Φ is an exact moment map for ω. (See Example 2.8).

There is a simple relationship between equivariant cohomology and Hamiltonian cobordism:

LEMMA 2.17. *Let $\omega_0 - \Phi_0$ and $\omega_1 - \Phi_1$ be equivariantly closed two-forms in the same equivariant cohomology class. Then the Hamiltonian G-manifolds (M, ω_0, Φ_0) and (M, ω_1, Φ_1) are cobordant.*

PROOF. Let μ be a one-form such that

$$(\omega_1 - \Phi_1) - (\omega_0 - \Phi_0) = (d + \iota(\xi_M))\mu.$$

Take the cobording manifold $M = [0, 1] \times M$ with the two-form $\omega_0 + d(t\mu)$ and moment map $\Phi^\xi = \Phi_0^\xi - t\mu(\xi_M)$. □

If we only assume that ω_0 and ω_1 are in the same ordinary de Rham cohomology class, we again obtain a cobordism, but only if we choose the moment maps carefully:

LEMMA 2.18. *Let ω_0 and ω_1 be closed two-forms on M in the same cohomology class. If the action is Hamiltonian for ω_0, it is also Hamiltonian for ω_1, and we can choose the moment maps Φ_0 and Φ_1 so that (M, ω_0, Φ_0) and (M, ω_1, Φ_1) are cobordant.*

PROOF. Suppose that $\omega_0 - \omega_1 = d\beta$. Let Φ_0 be a moment map for (M, ω_0). Then $\Phi_1^\xi = \omega_0 + \beta(\xi_M)$ is a moment map for (M, ω_1). They are cobordant through $W = [0, 1] \times M$ with $\omega = \omega_0 - d(t\beta)$ and $\Phi^\xi(t, m) = \Phi_0^\xi(m) + t\beta(\xi_M)$. □

The reason that we cannot specify the moment maps in advance in Lemma 2.18 is that in general a moment Φ is not cobordant to the moment map $\Phi + \alpha$ for $\alpha \in \mathfrak{g}^*$. If the two-form is symplectic, or, more generally, if $\int_M \omega^n \neq 0$, one can eliminate this ambiguity by normalizing moment maps as in [**GS5**]:

LEMMA 2.19. *Let M be a compact oriented $2n$-dimensional manifold with a G-action. Let ω_0 and ω_1 be closed two-forms in the same cohomology class such that $\int_M \omega_0^n = \int_M \omega_1^n \neq 0$. Let Φ_0 and Φ_1 be corresponding moment maps that satisfy the normalization condition*

$$\int_M \Phi_0 \omega_0^n = \int_M \Phi_1 \omega_1^n = 0. \tag{2.17}$$

Then (M, ω_0, Φ_0) and (M, ω_1, Φ_1) are cobordant.

PROOF. Consider the product $[0, 1] \times M$ with the two-form $\omega = \omega_0 - d(t\beta)$ and moment map $\Phi^\xi = \Phi_0^\xi + t\beta(\xi_M)$, where $d\beta = \omega_0 - \omega_1$. Denote by $i_t \colon M \to [0, 1] \times M$ the inclusion map $m \mapsto (t, m)$. Then $i_0^*\omega = \omega_0$, $i_1^*\omega = \omega_1$, and $i_0^*\Phi = \Phi_0$. It remains to show that $i_1^*\Phi = \Phi_1$. Since both of these are moment maps for ω_1, they differ by a constant, so it is enough to show that $\int_M (i_1^*\Phi)\omega_1^n = \int_M \Phi_1 \omega_1^n$. By assumption, the right term is zero. Because i_1 is homotopic to i_0, the left term is equal to $\int_M (i_1^*\Phi)\omega_1^n = \int_M i_1^*(\Phi\omega^n) = \int_M i_0^*(\Phi\omega^n) = \int_M \Phi_0 \omega_0^n = 0$. □

4. Proper Hamiltonian cobordisms

Let (M, ω, Φ) be a Hamiltonian G-manifold. Assume that the moment map $\Phi\colon M \to \mathfrak{g}^*$ is proper (the preimage of any compact set is compact); the manifold M need not be compact.

Many properties of compact Hamiltonian G-manifolds continue to hold for proper moment maps on non-compact manifolds. For instance, G is a torus and ω is symplectic, the moment image $\Phi(M)$ is a convex polyhedron; see [**CDM, LMTW**].

We would also like to refer to the Liouville measure on M and the Duistermaat–Heckman measure on $\mathfrak{g}^*$. However, the measure of an unbounded set might not be defined, because the set might be a union of two unbounded sets on which these measures are $+\infty$ and $-\infty$ respectively. To be more precise, one must define the Liouville measure and the Duistermaat–Heckman measure DH_M as *distributions.* Similarly, the Duistermaat–Heckman integral $I(\xi)$ can be replaced (up to sign) by the Fourier transform $\widehat{\mathrm{DH}}_M$. Concretely, the *Liouville distribution* and the *Duistermaat–Heckman distribution* are the continuous linear functionals

$$f \mapsto \int_M f\,\omega^n/n! \quad \text{ and } \quad \varphi \mapsto \int_{\mathfrak{g}^*} \varphi\,\mathrm{DH}_M = \int_M (\varphi\circ\Phi)\,\omega^n/n!$$

on the spaces $C_c^\infty(M)$ and $C_c^\infty(\mathfrak{g}^*)$ of compactly supported smooth functions. The Fourier transform $\widehat{\mathrm{DH}}_M$ associates to a function S on $\mathfrak{g}$ the value

$$\int_{\mathfrak{g}^*} \widehat{S}(\alpha)\,\mathrm{DH}_M = \int_{\mathfrak{g}^*}\int_{\mathfrak{g}} S(\xi)e^{-i\langle\alpha,\xi\rangle}d\xi\,\mathrm{DH}_M = \text{“}\int_g S(\xi)I(-\xi)d\xi\text{”}.$$

This is a distribution (in fact, a tempered distribution) if DH_M is a tempered distribution (see [**Rudi**]). This is the case if M is tame (diffeomorphic to the interior of a G-manifold with boundary), because DH_M is then “piecewise polynomial with finitely many pieces”.

As in the compact case, if G is a torus and Φ is proper, the Duistermaat–Heckman measure has a piecewise polynomial density function with respect to Lebesgue measure on $\mathfrak{g}^*$; see [**Pr, PW**]. The Duistermaat–Heckman formula, expressing the Duistermaat–Heckman integral in terms of fixed point data, holds under conditions that are slightly more restrictive than the moment map being proper: first, the fixed point set must be finite. Second, some component Φ^η of the moment map must be proper and bounded from below. Such a moment map is called *polarized* and is automatically proper. The generalization of the Duistermaat–Heckman formula to this case is due to Prato and Wu [**PW**]. Polarized moment maps play a major role in our theory, particularly in the linearization theorem of Chapter 4.

This discussion leads us to the definition of proper Hamiltonian cobordism:

Definition 2.20. A *proper Hamiltonian cobordism* between two Hamiltonian G-manifolds with proper moment maps, (M_0, ω_0, Φ_0) and (M_1, ω_1, Φ_1), is a Hamiltonian G-manifold (W, ω, Φ), where W is a manifold with boundary, and an orientation preserving equivariant diffeomorphism from $-M_0 \sqcup M_1$ to the boundary ∂W that carries ω to $\omega_0 \sqcup \omega_1$ and Φ to $\Phi_0 \sqcup \Phi_1$. Here, $-M_0$ denotes the manifold M_0 equipped with the opposite orientation.

The properness assumption on the moment maps is crucial; its omission can have disastrous consequences. For example, every manifold M is cobordant to the empty manifold via the cobordism $W = (0, 1] \times M$. It is easy to make this a

Hamiltonian cobordism if M is a Hamiltonian manifold, but the cobording moment map $(t, m) \mapsto \Phi(m)$ will not be proper.

LEMMA 2.21. *Proper Hamiltonian cobordism is an equivalence relation.*

PROOF. The only non-obvious part of this assertion is that proper Hamiltonian cobordisms are transitive. In other words, assume that (M_i, ω_i, Φ_i), for $i = 1, 2, 3$, are proper Hamiltonian G-manifolds, such that the first is properly cobordant to the second, and the second is properly cobordant to the third. Then the first is properly cobordant to the third.

This can be shown as follows. Let (W, ω, Φ) be a proper cobordism between the first and the second manifold and (W', ω', Φ') between the second and the third. We first modify these cobordisms so as to make them cylindrical near the boundary piece M_2, and then attach W to W' along this boundary piece to obtain a proper Hamiltonian cobordism from M_1 to M_3.

Being cylindrical means that a neighborhood of M_2 in W is isomorphic to a product $M_2 \times (-\epsilon, 0]$, where G only acts on the M_2 factor and where the two-form and moment map are the pull-backs from M_2.

A tubular neighborhood of M_2 in W is equivariantly diffeomorphic to $U = (-1, 0] \times M_2$. The proof of this in the non-equivariant case (see [**BJ**], [**MiSt**], or [**La1**]) goes through in the presence of a proper group action.

Consider the projection map $\pi \colon U \to M_2$ and the inclusion map $i \colon M_2 = \partial U \to U$. Because ω and Φ coincide with $\pi^*\omega_2$ and $\pi^*\Phi_2$ on ∂U, which is a strong deformation retract of U, there exists an invariant one-form β such that

$$\omega|_U = \pi^*\omega_2 + d\beta \quad \text{and} \quad \Phi|_U = \pi^*\Phi_2^\xi - \beta(\xi_W).$$

Let V be an open neighborhood of ∂U which is contained in $(-1/2, 0]$ and on which the difference between Φ and $\pi^*\Phi_2$ is, say, less than one (with respect to some metric on $\mathfrak{g}^*$). Let $\rho \colon U \to [0, 1]$ be a smooth function such that $\rho = 1$ outside V and $\rho = 0$ on some neighborhood of ∂U. The two-form $\pi^*\omega_2 + d(\rho(t)\beta)$ and moment map $\pi^*\Phi_2^\xi - \rho(t)\beta(\xi_W)$ extend to W and provide a proper Hamiltonian cobordism as required. $\square$

One may also work with *η-polarized Hamiltonian cobordisms*, meaning that the η-component of all moment maps is polarized (proper and bounded from below), for some fixed $\eta \in \mathfrak{g}$. Such cobordism is also an equivalence relation; this is proved exactly like Lemma 2.21.

However, the composition of an η-polarized cobordism and an η'-polarized cobordism might not be polarized if $\eta \neq \eta'$. (As a trivial counterexample with $G = S^1$, a disjoint union $W \sqcup W'$ can be viewed as a composition of cobordisms along an empty boundary and is not polarized if the moment map is unbounded from above on W and unbounded from below on W'.)

In the rest of this section we assume that G is a torus.

REMARK 2.22 (Locality of Hamiltonian cobordisms). We may fix a convex open subset U of $\mathfrak{g}^*$ and assume that the moment image is contained in U and that the moment map is proper as a map to U. The convexity, connectedness, and stability theorems continue to hold; see [**LMTW**]. We can also work with *proper Hamiltonian cobordisms over U*. An interesting observation is that cobordism is a local notion: if two Hamiltonian spaces are properly cobordant over U and over V, then they are properly cobordant over $U \cup V$ (possibly through an orbifold).

The idea of the proof is to "cut" (a la Lerman [**Ler1**]) each manifold into pieces, each of which maps properly to U or V, and note that the result of the cutting is cobordant to the original manifold.

Allowing proper Hamiltonian cobordisms enriches the compact theory with new objects without degenerating it with any new equivalences:

THEOREM 2.23. *Suppose that two Hamiltonian manifolds are compact and are properly cobordant. Then they are also compactly cobordant (possibly through an orbifold).*

This is proved in [**GGK2**] for abstract moment maps (cf. Remark 3.29). A similar proof holds for Hamiltonian manifolds. Hence, the theory of proper Hamiltonian cobordisms is at least as non-trivial as the theory of compact Hamiltonian cobordisms. Non-triviality is also illustrated by the following result (cf. Lemma 2.13):

THEOREM 2.24. *Two Hamiltonian G-manifolds that are properly cobordant have the same Duistermaat–Heckman measure.*

PROOF. Let (W, ω, Φ) be a proper Hamiltonian cobordism between (M_0, ω_0, Φ_0) and (M_1, ω_1, Φ_1). It is enough to prove that for any compactly supported smooth function $\varphi\colon \mathfrak{g}^* \to \mathbb{R}$, the integrals of φ against the two Duistermaat–Heckman measures are equal; equivalently, that

$$\int_{M_0} (\varphi \circ \Phi_0)\, \omega_0^n/n! = \int_{M_1} (\varphi \circ \Phi_1)\, \omega_1^n/n!.$$

This equality follows from Stokes' theorem, once we notice that the integrands extend to the differential form $(\varphi \circ \Phi)\omega^n$ on the coboring manifold W, and that this form is closed (by Lemma 2.14) and compactly supported. □

5. Hamiltonian complex cobordisms

The notion of Hamiltonian cobordism captures invariants which are determined by the equivariant cohomology class $[\omega - \Phi]$, such as the Duistermaat–Heckman measure. Other invariants, like the geometric quantization considered in Chapter 6, require a richer structure. There are various additional structures which enable one to define geometric quantization. We will work with *stable complex Hamiltonian G-manifolds*, i.e., Hamiltonian G-manifolds equipped with G-equivariant stable complex structures.[1] Recall that a stable complex structure is given by a complex structure J on the fibers of a Whitney sum $TM \oplus \mathbb{R}^k$; see Section 1 of Appendix D for details.

The manifolds that we care to quantize are *symplectic* manifolds, to which there are naturally associated *almost complex* structures. (See Example D.12.) However, we choose to work with stable complex, not just almost complex structures, for several reasons.

One advantage of stable complex structures is that one can consider their cobordisms. This is in contrast to almost complex structures, which can only exist on even dimensional manifolds.

[1] There are strong arguments in favor of working with a slightly different structure, namely, a Spin^c-structure, in order to define geometric quantization. See [**CKT**], [**Par3**], or Section 3.5 of Appendix D.

DEFINITION 2.25. A *Hamiltonian complex cobordism* between two stable complex Hamiltonian G-manifolds is a Hamiltonian cobordism between these manifolds, together with a G-equivariant stable complex structure on the cobording manifold which restricts to the given structures on the boundary.

For a more precise formulation, see Proposition D.14 and Definition D.22.

As before, we may consider compact cobordisms, proper cobordisms, or η-polarized cobordisms, and each of these is an equivalence relations.

The geometric quantization for an "integral" stable complex Hamiltonian G-manifolds, (defined in Section 7 of Chapter 6,) is invariant under Hamiltonian complex G-cobordism. See Section 7 of Chapter 6 and Appendix J.

Another motivation for considering stable complex structures comes from the notion of *reduction.* A G-invariant almost complex structure which is incompatible with a symplectic form does not descend to an almost complex structure on the reduced space $\Phi^{-1}(\alpha)/G$, but only to a stable complex structure. (See Section 2.3 of Chapter 5.) Incompatible almost complex structures arise naturally in the cobordism linearization theorem (see Chapter 4) and in its quantum version (see Chapter 7). See also [**GGK1**].

Finally, equivariant stable complex structures enable one to "resolve" the orbifold singularities of a reduced space; cf. [**GGK2**]. In this case, the "locality of Hamiltonian cobordisms" (Remark 2.22) and the fact that "compact cobordisms classes inject into proper cobordism classes" (Theorem 2.23) are true without introducing orbifolds.