CHAPTER 3

Abstract moment maps

Many aspects of Hamiltonian group actions involve the moment map and not the two-form. Hence, it is often convenient to focus entirely on the properties of moment maps and to ignore two-forms. For this purpose we introduce the notion of an *abstract moment map*.

1. Abstract moment maps: definitions and examples

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual space. For any map Ψ which takes values in $\mathfrak{g}^*$ and any subgroup H of G with Lie algebra $\mathfrak{h}$ we denote by $\Psi^{\mathfrak{h}}$ or Ψ^H the composition of Ψ with the natural projection $\mathfrak{g}^* \to \mathfrak{h}^*$. Similarly, for any Lie algebra element $\xi \in \mathfrak{g}$, we denote by Ψ^ξ the ξth component of Ψ, i.e., the real valued function $\langle \Psi, \xi \rangle$.

Definition 3.1 ([**Ka3**]). Let M be a G-manifold. An *abstract moment map* on M is a smooth map $\Psi \colon M \to \mathfrak{g}^*$ with the following two properties:

1. Ψ is G-equivariant;
2. For any subgroup H of G, the map $\Psi^H \colon M \to \mathfrak{h}^*$ is locally constant on the submanifold M^H of points fixed by H.

The values of Ψ^H on the connected components of M^H form the *assignment* associated with Ψ; see Appendix E.

The second requirement in Definition 3.1 can be replaced by

(2') For any Lie algebra element $\xi \in \mathfrak{g}$, the function Ψ^ξ is locally constant on the zero set M^ξ of the corresponding vector field ξ_M.

Example 3.2. The constant function zero is an abstract moment map.

Example 3.3. On a manifold with a circle action with isolated fixed points, any invariant function is an abstract moment map.

Example 3.4. Let X be a vector field on M that generates a circle action and let $\langle , \rangle$ be an invariant Riemannian metric on M. Then the function $\langle X, X \rangle$ is an abstract moment map with zero assignment.

Example 3.5. If $\Psi \colon M \to \mathfrak{g}^*$ is an abstract moment map and $\alpha \in \mathfrak{g}^*$ is fixed by the coadjoint action, the translation $\Psi + \alpha$ is also an abstract moment map. In the special case $G = S^1$, the composition $F \circ \Psi$ is an abstract moment map for any smooth $F \colon \mathbb{R} \to \mathbb{R}$.

Example 3.6. Let M be a G-manifold and let $\Psi \colon M \to \mathfrak{g}^*$ be a moment map for an invariant closed two-form ω on M. Recall, that this means Ψ is G-equivariant and satisfies Hamilton's equation,

$$\iota(\xi_M)\omega = d\Psi^\xi \quad \text{for all } \xi \in \mathfrak{g}. \tag{3.1}$$

Then Ψ is an abstract moment map. Indeed, the coordinates of Φ^H are Φ^ξ for $\xi \in \mathfrak{h}$. These satisfy $d\Phi^\xi|_{M^H} = \iota(\xi_M)\omega|_{M^H} = 0$ because ξ_M vanishes on M^H.

If equation (3.1) holds, we say that ω is compatible with Ψ, or that Ψ is associated with ω. An abstract moment map associated with some two-form is called a *Hamiltonian* (abstract) moment map.

Not every abstract moment map is Hamiltonian. See Example E.36. In Theorem E.37 we give a condition for an abstract moment map to be Hamiltonian. Moment maps associated with true symplectic forms must additionally satisfy some non-degeneracy requirements, which we analyze in Appendix G.

It is worth pointing out that a moment map on a Poisson manifold (see, e.g., [**CW**]) might not be an abstract moment map even when the Poisson structure is preserved by the action. The reason is that, since a moment map is defined only up to addition of Casimir functions, and since on a Poisson manifold Casimir functions often exist in abundance, a moment map on a Poisson manifold might not be constant on the fixed point set.

EXAMPLE 3.7. Let a Lie group G act on a manifold M and let μ be any invariant one-form. Then the function $\Psi\colon M \to \mathfrak{g}^*$ defined by

$$\Psi^\xi = \mu(\xi_M) \tag{3.2}$$

is an abstract moment map.

Such moment maps have been studied on contact manifolds M, when μ is a contact form, in [**Al, Ge, Wt, LW, Lo**], specifically in the context of reduction. A definition of a contact moment map which is independent of a choice of contact one-form was given by Lerman in [**Ler3**]:

EXAMPLE 3.8. Following [**Ar**, Appendix 4] and [**Ler3**], we consider a contact manifold (M, ξ) with a G-action preserving the contact structure ξ. Recall that ξ is a maximally non-integrable codimension one sub-bundle of the tangent bundle TM. Let ξ^0 be its annihilator in T^*M. The corresponding *contact moment map* is the exact moment map on ξ^0 corresponding to the canonical one-form

$$\Psi\colon \xi^0 \to \mathfrak{g}^*, \qquad \Psi^\xi(q,p) = p(\xi_M|_q).$$

More generally, an abstract moment map that arises by equation (3.2) is called *exact* (cf. Example 2.8). A compatible two-form is then given by $\omega = -d\mu$. An exact moment map has the property that Ψ^H vanishes on M^H for all $H \subseteq G$. This property *characterizes* exact moment maps among all abstract moment maps; see Corollary E.27. Many "classical" moment maps, such as the canonical moment map on a cotangent bundle, or the moment map on a pre-quantization circle bundle, are exact. See Examples 3.11 and 3.12.

The following two examples exhibit functoriality properties of abstract moment maps with respect to G and M, respectively.

EXAMPLE 3.9. Let M be a G-manifold with an abstract moment map $\Psi\colon M \to \mathfrak{g}^*$. Let a Lie group H act on M through a group homomorphism $\varphi\colon H \to G$ followed by the G-action. Then the composition of Ψ with the natural map $\varphi^*\colon \mathfrak{g}^* \to \mathfrak{h}^*$ is an abstract moment map for the H-action. If Ψ is a Hamiltonian, respectively, exact, moment map, so is $\varphi^* \circ \Psi$.

EXAMPLE 3.10. Let M be a G-manifold with an abstract moment map $\Psi\colon M \to \mathfrak{g}^*$. Let N be another G-manifold and let $f\colon N \to M$ be a G-equivariant map.

Then the pullback $f^*\Psi = \Psi \circ f$ is an abstract moment map for the G-action on N. This map is proper if f and Ψ are proper. If Ψ is a Hamiltonian, respectively, exact, moment map, so is $f^*\Psi$.

EXAMPLE 3.11. Following [**SLM**] and [**Lew**], consider a G-manifold Q, and denote by $J\colon T^*Q \to \mathfrak{g}^*$ the canonical moment map: $J^\xi(\alpha) = \alpha(\xi_Q(x))$, where $\alpha \in T^*_x Q$. The action map $F\colon Q \times \mathfrak{g} \to TQ$ is defined by $F(x,\xi) = \xi_Q(x)$ (cf. Section 1.6 of Appendix B). Consider a Lagrangian on Q with Legendre transformation $\mathbb{L}\colon TQ \to T^*Q$; see, e.g., [**Ar**]. By Example 3.10, the composition

$$\mathbb{I} = (\mathbb{L}F)^*J\colon Q \times \mathfrak{g} \xrightarrow{F} TQ \xrightarrow{\mathbb{L}} T^*Q \xrightarrow{J} \mathfrak{g}^*$$

is an exact abstract moment map. For example, assume that $\mathbb{L}$ arises from a Riemannian metric $\langle\ ,\ \rangle$ on Q so that $\mathbb{L}(v) = \langle v, \cdot\rangle$ for a tangent vector v. Then $\mathbb{I}^\zeta(x,\xi) = \langle \xi_Q(x), \zeta_Q(x)\rangle$ for all $\zeta \in \mathfrak{g}$. The map $\mathbb{I}$, called the locked inertia tensor, is used in the analysis of relative equilibria. (See [**SLM**] and [**Lew**].) Note that in general $Q \times \mathfrak{g}$ is not a symplectic manifold and G, in this example, does not have to be commutative.

EXAMPLE 3.12. Let (M, ω, Ψ) be a Hamiltonian G-manifold. Let $\pi\colon P \to M$ be a pre-quantization circle bundle. This means that P is a G-equivariant principal circle bundle with a connection one-form Θ, such that $d\Theta = -\pi^*\omega$, and such that the moment map Ψ is given by $\pi^*\Psi^\xi = \Theta(\xi_P)$ for all $\xi \in \mathfrak{g}$. (For more details, see Chapter 6.) Then the pullback $\pi^*\Psi\colon P \to \mathfrak{g}^*$ is an exact moment map.

EXAMPLE 3.13. A linear combination of abstract moment maps is an abstract moment map.

The advantage of working with exact moment maps over Hamiltonian ones is that these can be patched together:

EXAMPLE 3.14. If Ψ_0 and Ψ_1 are exact moment maps, then so is $(1-\rho)\Psi_0 + \rho\Psi_1$ for any smooth function $\rho\colon M \to [0,1]$. More generally, if Ψ_0 and Ψ_1 arise from one-forms μ_0 and μ_1, then $f\Psi_0 + g\Psi_1$ arises from the one-form $f\mu_0 + g\mu_1$, for any smooth functions f and g.

EXAMPLE 3.15. A direct sum of abstract moment maps on the same manifold is *not* necessarily an abstract moment map. Namely, suppose that G_1 and G_2 act on M with abstract moment maps Ψ_1 and Ψ_2, that the two actions commute with each other, and that Ψ_1 and Ψ_2 are invariant under both actions. The sum $\Psi_1 \oplus \Psi_2$ is not necessarily an abstract moment map for the $G_1 \times G_2$-action as it would be if Ψ_1 and Ψ_2 were associated with the same closed two-form. For example, let $G_1 = G_2$ be the circle and assume that the G_1-action coincides with the G_2-action. Then the anti-diagonal $H \subset G_1 \times G_2$ acts trivially on M. The H-component of $\Psi_1 \oplus \Psi_2$ is just $\Psi_1 - \Psi_2$. The direct sum is an abstract moment map if and only if $\Psi_1 = \Psi_2 + \mathit{const}$.

2. Proper abstract moment maps

In this section we focus on abstract moment maps that are *proper*. Recall, a map is *proper* if the preimage of any compact set is compact.

EXAMPLE 3.16. The zero map is a proper abstract moment map if and only if the manifold is compact.

EXAMPLE 3.17. If the fixed point set M^G has a non-compact component, M does not admit a proper abstract moment map.

EXAMPLE 3.18. Let V be a complex vector space with a linear circle action that fixes only the origin. For any invariant Hermitian inner product, $\langle \cdot, \cdot \rangle$, the function $\Psi(v) = \langle v, v \rangle$ is an abstract moment map. It is proper if and only if $\langle \cdot, \cdot \rangle$ is positive or negative definite.

In particular, the moment map $\frac{1}{2} \sum m_j |z_j|^2$ for the Hamiltonian circle action on $\mathbb{C}^n$ with weights $-m_j \in \mathbb{Z}$ is proper if and only if all the m_j's are positive or all are negative.

More generally, for a torus action on $\mathbb{C}^n$ with weights $-\alpha_j \in \mathbb{Z}_G^*$, the moment map $\frac{1}{2} \sum |z_j|^2 \alpha_j$ is proper if and only if there exists $\eta \in \mathfrak{g}$ such that $\alpha_j(\eta) > 0$ for all j. See Proposition 4.15.

DEFINITION 3.19. Let $\eta \in \mathfrak{g}$ be a Lie algebra element. A function $\Psi \colon M \to \mathfrak{g}^*$ is said to be *η-polarized* if its ηth component, $\Psi^\eta \colon M \to \mathbb{R}$, is proper and bounded from below. It is *polarized* if it is η-polarized for some η.

A polarized function is always proper, but a proper function (e.g., $\mathrm{id} \colon \mathfrak{g}^* \to \mathfrak{g}^*$) is not necessarily polarized. If M is compact, all functions $M \to \mathfrak{g}^*$ are η-polarized for all $\eta \in \mathfrak{g}$.

A linear combination of proper abstract moment maps is an abstract moment map, but it is not necessarily proper. However, a positive linear combination of η-polarized abstract moment maps is again an η-polarized abstract moment map. This technical fact plays an important role in our theory and is the key reason for the introduction of polarized abstract moment maps.

More specifically, polarized abstract moment maps possess the following two properties, which may in generally fail for proper abstract moment maps:

- A positive linear combination of η-polarized abstract moment maps on the same manifold is again an η-polarized (hence proper) abstract moment map.
- Let $\Psi_j \colon M_j \to \mathfrak{g}^*$, $j = 1, 2$, be η-polarized abstract moment maps. Consider the product $M_1 \times M_2$ with the diagonal G-action. Then $\Phi(m_1, m_2) := \Phi_1(m_1) + \Phi_2(m_2)$ is an η-polarized (hence proper) abstract moment map.

The first of these two properties allows us to carry out "patching" arguments. (See Appendix E.) The second of the two properties gives a ring structure on the cobordism classes of η-polarized abstract moment maps.

3. Cobordism

Recall, a *cobordism* between two manifolds, M_0 and M_1, is a manifold with boundary W, and a diffeomorphism from the disjoint union of M_0 and M_1 to the boundary of W:

$$\partial W = M_0 \sqcup M_1. \tag{3.3}$$

For oriented manifolds, one insists that boundary orientation on ∂W transports to the given orientation on M_1 and the opposite orientation on M_0.

In ordinary cobordism theory, one only considers compact manifolds; otherwise, the theory becomes trivial: every manifold M is non-compactly cobordant to the empty set via $W = (0, 1] \times M$.

An *equivariant* cobordism is one that carries a group action, i.e., a cobordism in which M_0, M_1, and W are G-manifolds, and the diffeomorphism (3.3) is equivariant.

EXAMPLE 3.20. A compact manifold that admits a free S^1-action is (equivariantly) cobordant to the empty set. Indeed, such a manifold, M, is a circle bundle. Take the associated disc bundle $W = M \times_{S^1} D^2$. It still admits an S^1-action because S^1 is abelian. Its boundary is diffeomorphic to M via $m \mapsto [m, 1]$. Notice, however, that the fixed point set W^{S^1} is always non-empty, as it contains the zero section M/S^1. In fact, if the action on M is free, M may fail to be cobordant to the empty set in the class of manifolds with free S^1 actions. This is already the case when $M = S^1$, as follows from the fact that $M/S^1 =$ point is not the boundary of any compact manifold.

We now consider cobordisms between abstract moment maps. We do not require our manifolds to be compact; the compactness assumption is replaced by the demand that the abstract moment maps be proper. In what follows, all abstract moment maps will be proper unless otherwise specified.

DEFINITION 3.21. Let M_0 and M_1 be oriented G-manifolds with proper abstract moment maps $\Psi_0 \colon M_0 \to \mathfrak{g}^*$ and $\Psi_1 \colon M_1 \to \mathfrak{g}^*$. A *proper cobordism* between (M_0, Ψ_0) and (M_1, Ψ_1) is a (possibly non-compact) equivariant oriented cobordism W between M_0 and M_1, and a proper abstract moment map $\Psi \colon W \to \mathfrak{g}^*$, such that the equivariant diffeomorphisms (3.3) carries the abstract moment map on W to those on M_0 and M_1:

$$\Psi|_{\partial W} = \Psi_0 \sqcup \Psi_1.$$

REMARK 3.22. We may refer to the above as a cobordism between M_0 and M_1 if it is clear from the context what moment maps on M_0 and M_1 we consider. For instance, on a proper Hamiltonian G-manifold (M, ω, Φ) (see Chapter 2) we take the moment map Φ. A compact G-manifold is by default equipped with the zero moment map unless specified otherwise.

EXAMPLE 3.23. Every compact equivariant cobordism becomes a proper cobordism by setting all abstract moment maps to be identically zero.

EXAMPLE 3.24. A compact manifold M with a free S^1-action is properly cobordant to the empty set via $W = (0, 1] \times M$ and $\Psi(t, m) = \frac{1-t}{t}$.

It is absolutely crucial for the abstract moment maps to be proper; otherwise, every manifold M would be cobordant to the empty set via the non-compact cobordism $M \times (0, 1]$, where the G action and abstract moment maps are induced from those on M. It is also crucial to assume Condition 2 of Definition 3.1, that the abstract moment map components must be locally constant on appropriate fixed point sets. Dropping this condition would result in new identifications; for instance, every compact S^1-manifold would then be cobordant to the empty set via the non-compact cobordism $(0, 1] \times M$ and the proper function $(t, m) = \Phi(m) + \frac{1-t}{t}$.

LEMMA 3.25. *Cobordism of proper abstract moment maps is an equivalence relation.*

PROOF. As in the proof of Lemma 2.21, the crucial point is to modify a cobording G-manifold and proper abstract moment map $\Psi \colon W \to \mathfrak{g}^*$ so as to make it cylindrical near a connected component M_2 of the boundary ∂W. A tubular neighborhood of M_2 is equivariantly diffeomorphic to $U = (-1, 0] \times \partial M_2$; cf. [**MiSt**]. Let Ψ_2 denote the restriction of Ψ to M_2. Let $\pi\colon U \to M_2$ denote the projection map. Let V be a neighborhood of ∂U which is contained in $(-1/2, 0]$ and on which the difference between Ψ and $\pi^*\Psi_2$ is, say, less than one (with respect to some metric on $\mathfrak{g}^*$). Let $\rho\colon U \to [0, 1]$ be a smooth function such that $\rho = 1$ outside V and $\rho = 0$ on some neighborhood of the boundary M_2. Then $\pi^*\partial\Psi + \rho(t)(\Psi - \pi^*\partial\Psi)$ extends to a proper abstract moment map on W which is cylindrical near M_2. □

REMARK 3.26. Various extra structures can be incorporated in the notion of cobordism of abstract moment maps. For example, may consider stable complex cobordisms (see Section 5 of Chapter 2). Finally, the assumption that all abstract moment maps be proper can be replaced by a stronger requirement that they be η-polarized with respect to a fixed vector $\eta \in \mathfrak{g}$. Note that η-polarized cobordism is also an equivalence relation; the proof of Lemma 3.25 still goes through.

EXAMPLE 3.27. A proper Hamiltonian cobordism becomes a cobordism of proper abstract moment maps once the two-forms are discarded. (See Definition 2.20.)

The relationship between Hamiltonian cobordisms and cobordisms of abstract moment maps is discussed in Section 5 of Appendix E.

Let us now turn to the relationship between proper cobordism and ordinary compact equivariant cobordism.

The following example shows that we do not introduce new relations between *non-equivariant* compact cobordism classes by taking their *equivariant non-compact* cobordism classes with zero moment maps:

EXAMPLE 3.28. Let M_0 and M_1 be compact manifolds with trivial G-actions. Let W be a G-manifold with boundary $\partial W = M_0 \sqcup M_1$ and a proper abstract moment map $\Psi\colon W \to \mathfrak{g}^*$. Then M_0 and M_1 are compactly cobordant. Indeed, because G acts trivially on ∂W, it acts trivially on its neighborhood, and hence it acts trivially everywhere, after discarding any connected components which are disjoint from ∂W. A G-manifold with a trivial action and proper moment map must be compact, because its moment map must be constant.

Any proper cobordism of manifolds with trivial G-actions and abstract moment maps is automatically compact. Indeed, a connected group acts on a connected manifold with boundary and the action is trivial on the boundary, (or, in fact, on any submanifold of codimension one), it acts trivially everywhere. A manifold with a trivial action and a proper abstract moment map must be compact, because its moment map must be constant.

REMARK 3.29 (Compact versus non-compact cobordism). If two *compact* G-manifolds with (possibly zero) abstract moment maps are properly cobordant through a non-compact manifold W, and if W admits an equivariant stable complex structure, then the two manifolds are also cobordant via some other, compact, W'. The way to show this is to first use Lerman's cutting procedure [**Ler1**] to replace a non-compact cobordism by a compact orbifold cobordism, and then to get rid of the orbifold singularities by successive equivariant surgery along singular strata. The details of this argument can be found in [**GGK2**]. If the group G is the circle group, the presence of a stable complex structure can be replaced by a restriction on the stabilizer subgroups which are allowed to occur in the cobordism; see [**GGK2**].

Finally, we note that the set of proper cobordism classes of oriented G-manifolds equipped with proper abstract moment maps forms an abelian group under the operation of disjoint unions. Moreover, for any Lie algebra element $\eta \in \mathfrak{g}$, the set of η-polarized cobordism classes forms a *ring*. This follows from the fact, noted in Section 2, that if M_1 and M_2 are equipped with η-polarized abstract moment maps, so is $M_1 \times M_2$.

4. First examples of proper cobordisms

The following *uniqueness lemmas* assert that an abstract moment map is determined by its values at the fixed points and its behavior at infinity. We restrict our attention to circle actions, and, first to linear circle actions:

LEMMA 3.30. *Let $M = V$ be a vector space with a linear circle action. Let Ψ_0 and Ψ_1 be two proper abstract moment maps on V which take the same value at the origin and such that $\Psi_0(v)$ and $\Psi_1(v)$ either both approach ∞ or both approach $-\infty$ as $\|v\|$ goes to ∞. Then Ψ_0 and Ψ_1 are properly cobordant.*

We note that a proper abstract moment map $\Psi \colon V \to \mathbb{R}$ must satisfy either $\Psi(v) \to \infty$ or $\Psi(v) \to -\infty$ as $\|v\| \to \infty$. Indeed, the preimage of the closed interval $[-N, N]$ is contained in some large ball B_N in V; the complement $V \smallsetminus B_N$ maps to either $[N, \infty]$ or $[N, -\infty]$, because it is connected.

We also note that the sub-space of fixed points is compact (because Ψ is constant on it and is proper), and hence equal to the origin: $V^{S^1} = \{0\}$.

More generally,

LEMMA 3.31. *Let Ψ_0 and Ψ_1 be proper abstract moment maps on a manifold M with a circle action. Suppose that Ψ_0 and Ψ_1 take the same values at the fixed points and are both bounded from below or both bounded from above. Then Ψ_0 is properly cobordant to Ψ_1.*

Yet more generally, if M is a *tame* S^1-manifold (equivariantly diffeomorphic to the interior of a compact S^1-manifold with boundary), two proper abstract moment maps are properly cobordant if they take the same values at the fixed points and, at each end of M, either both approach $+\infty$ or both approach $-\infty$. Note that since M is a two-dimensional S^1-manifold the requirement that M is tame is equivalent to that M has a finite number of connected components.

PROOFS. Take the trivial cobordism $[0, 1] \times M$ with the abstract moment map $\Psi(t, m) = (1 - t)\Psi_0(m) + t\Psi_1(m)$. □

EXAMPLE 3.32. Consider the plane $\mathbb{C} = \mathbb{R}^2$ with the standard S^1-action and the standard orientation. We denote by $\mathbb{C}(a, +\infty)$ the cobordism class of an abstract moment map Ψ which takes the value a at the origin and which is bounded from below. This is well defined by Lemma 3.30. Similarly, we denote by $\mathbb{C}(a, -\infty)$ the cobordism class of the plane with the standard action and orientation and a proper abstract moment map which takes the value a at the origin and which is bounded from above. For instance, we may assume that the moment maps on $\mathbb{C}(a, \pm\infty)$ are just $\Psi(z) = a \pm |z|^2$, $z \in \mathbb{C}$.

Let $\overline{\mathbb{C}}(a, \pm\infty)$ denote the space $\mathbb{C}(a, \pm\infty)$ with the opposite circle action; let $-\mathbb{C}(a, \pm\infty)$ denote $\mathbb{C}(a, \pm\infty)$ with the opposite orientation. Then

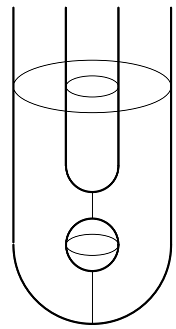

FIGURE 3.1. Linearization Theorem: S^2 is cobordant to two planes.

$$\overline{\mathbb{C}}(a, \pm\infty) = -\mathbb{C}(a, \pm\infty) \tag{3.4}$$

via complex conjugation: $z \mapsto \overline{z}$.

EXAMPLE 3.33. Denote by $S^2(a, b)$ the cobordism class of the sphere $S^2 \subset \mathbb{R}^3$ with the standard orientation, the standard circle action by rotations about the z-axis, and an abstract moment map Ψ which takes the values a and b on the north and south poles. (All such Ψ's are cobordant by Lemma 3.31.) Then

$$S^2(a, b) = \mathbb{C}(a, +\infty) + \overline{\mathbb{C}}(b, +\infty). \tag{3.5}$$

The cobordism giving rise to (3.5) is shown in Figure 3.1. As the cobording abstract moment map we can take any function $\tilde{\Psi}$ that satisfies the following conditions:

- $\tilde{\Psi}$ is proper and bounded from below. For example, on the complement of a sufficiently large ball we may take $\tilde{\Psi}$ to be the projection to the z-axis.
- $\tilde{\Psi}$ takes the values a and b at the north and south poles, respectively.
- $\tilde{\Psi}$ is constant on the segments of the z-axis contained in the cobording manifold. (This is necessary because these segments form the fixed point set for the S^1-action.)

The restrictions of this function to S^2 and to the two copies of $\mathbb{C}$ are cobordant to the given abstract moment maps on these spaces by Lemmas 3.30 and 3.31.

The cobordism (3.5) is the first example of the Linearization Theorem: the sphere is properly cobordant to the disjoint union of its tangent planes at the fixed points, equipped with the induced orientations and S^1-actions.

In a similar way, we have

$$S^2(a, b) = \mathbb{C}(a, -\infty) + \overline{\mathbb{C}}(b, -\infty), \tag{3.6}$$

which in combination with (3.5) implies the relation

$$\mathbb{C}(a, +\infty) + \overline{\mathbb{C}}(b, +\infty) = \mathbb{C}(a, -\infty) + \overline{\mathbb{C}}(b, -\infty). \tag{3.7}$$

EXAMPLE 3.34. Consider the cylinder $C = S^1 \times \mathbb{R}$ with coordinates $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and t, the standard orientation, given by $d\theta \wedge dt$, and the standard S^1-action, generated by $\partial/\partial\theta$. Consider the following proper abstract moment maps:

- $\Psi_+(e^{i\theta}, t) = t$;
- $\Psi_-(e^{i\theta}, t) = -t$;
- $\Psi_1(e^{i\theta}, t) = t^2$;

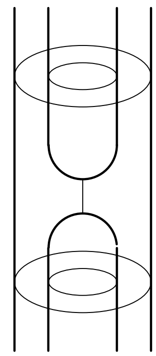

FIGURE 3.2. A cylinder is cobordant to two planes.

- $\Psi_2(e^{i\theta}, t) = -t^2$.

We have

$$(C, \Psi_-) = -(C, \Psi_+),$$

where the minus sign denotes the opposite orientation, via $(e^{i\theta}, t) \mapsto (e^{i\theta}, -t)$. Both (C, Ψ_1) and (C, Ψ_2) are cobordant to the empty set via the trivial cobordism $[0, \infty) \times C$ with the abstract moment maps $\Psi(x, t) = \pm(x + t^2)$.

EXAMPLE 3.35. Consider the cylinder $C = S^1 \times \mathbb{R}$ with the standard S^1-action, the standard orientation, and the abstract moment map $\Psi_+(e^{i\theta}, t) = t$. Let C_+ denote its cobordism class. Then for any $a \in \mathbb{R}$,

$$C_+ = \overline{\mathbb{C}}(a, +\infty) + \mathbb{C}(a, -\infty). \tag{3.8}$$

A cobordism giving (3.8) is sketched in Figure 3.2. The moment map $\tilde{\Psi}$ on the cobordism can be taken to be equal to the projection to the z-axis on the complement to a sufficiently large ball. On the "middle part" of the cobordism we set $\tilde{\Psi} \equiv a$, so that $\tilde{\Psi}$ is constant on the fixed point set (the segment of the z-axis contained in the cobording manifold). The abstract moment map obtained by restricting $\tilde{\Psi}$ to C is cobordant to Ψ_+. An explicit construction of this cobordism (a formula rather than a picture) is given in Section 8 of Chapter 5.

Note that (3.8) implies that

$$\overline{\mathbb{C}}(a, +\infty) + \mathbb{C}(a, -\infty) = \overline{\mathbb{C}}(b, +\infty) + \mathbb{C}(b, -\infty), \tag{3.9}$$

which is equivalent to (3.7) by (3.4).

5. Cobordisms of surfaces

In this section we will compute the cobordism group of surfaces with S^1-actions and proper abstract moment maps.

We note that in an oriented surface with an effective S^1-action each orbit is a fixed point or the action on the orbit is free. This follows from the local normal form and the fact that an effective action is locally effective if the manifold is connected (see Appendix B). We also note that a connected surface S^1-manifold is automatically tame, meaning, equivariantly diffeomorphic to the interior of a compact S^1-manifold. This follows, by looking at M/S^1, from the local normal form and the classification of one-dimensional manifolds.

Recall from Example 3.32 that $\mathbb{C}(a, \pm\infty)$ denotes the cobordism class of $\mathbb{C} = \mathbb{R}^2$ with the standard orientation and circle action and with the moment map $\Psi(z) = a \pm |z|^2$.

THEOREM 3.36. *Let $G = S^1$ and let (M, Ψ) be a connected oriented surface with an effective G-action and a proper abstract moment. Then (M, Ψ) is properly cobordant to a finite (possibly empty) disjoint union of planes $\mathbb{C}(a, \pm\infty)$ and $\overline{\mathbb{C}}(a, \pm\infty)$.*

Let Γ_{eff} be the group of cobordism classes of (M, Ψ) where M is a tame oriented surface with a locally effective circle action and a proper abstract moment map. (Equivalently, M has finitely many connected component and the action is effective on each component.)

THEOREM 3.37. *The group Γ_{eff} is the abelian group generated by $\mathbb{C}(a, \pm\infty)$, $a \in \mathbb{R}$, with the relations generated by*

$$\mathbb{C}(a, +\infty) - \mathbb{C}(a, -\infty) = \mathbb{C}(b, +\infty) - \mathbb{C}(b, -\infty). \tag{3.10}$$

COROLLARY 3.38. The group Γ_{eff} is the free abelian group generated by the spaces $\mathbb{C}(a, +\infty)$, $a \in \mathbb{R}$, and $\mathbb{C}(0, -\infty)$. Alternatively, Γ_{eff} is the free abelian group generated by $\mathbb{C}(a, +\infty)$, $a \in \mathbb{R}$, and C_+. (See Example 3.35.)

PROOF OF THEOREM 3.36. *Step 1.* The following is the list of orientable (although not oriented) connected surfaces with effective circle actions:

- $\mathbb{T}^2 = (S^1)_1 \times (S^1)_2$ with the standard action by rotations on the first factor;
- S^2 with the standard action by rotations about the z-axis;
- $\mathbb{R}^2 = \mathbb{C}$ with the standard action by rotations about the origin;
- the cylinder $C = S^1 \times \mathbb{R}$ with the standard action by rotations on the first component.

EXERCISE. Prove that an orientable connected surface with an effective S^1-action is equivariantly diffeomorphic to one from the above list.

It is clear that if Theorem 3.36 holds for a surface with one orientation, it also holds for the same surface with the opposite orientation. It suffices therefore to prove that the theorem for G-surfaces from the above list equipped with the standard orientation and an arbitrary abstract moment map.

Step 2. An abstract moment map Ψ on $\mathbb{T}^2$ is just a smooth function on the second factor, $\mathbb{T}^2/G = (S^1)_2 \to \mathbb{R}$. Let $N = \mathbb{T}^2 \times_G D^2 = D^2 \times (S^1)_2$, where G acts on D^2 in the standard way. Extend Ψ to a smooth G-invariant function $\tilde{\Psi}$ on N which is constant on $N^G = \{0\} \times (S^1)_2$. The pair $(N, \tilde{\Psi})$, where N is oriented according to the orientation of $\mathbb{T}^2$, gives a cobordism between $(\mathbb{T}^2, \Psi)$ and zero.

Step 3. Let Ψ be an abstract moment map on S^2, i.e., Ψ is just a G-invariant function. As shown in Example 3.33, (S^2, Ψ) is cobordant to $\mathbb{C}(a, +\infty) + \overline{\mathbb{C}}(b, +\infty)$, where a and b are the values of Ψ at the north and the south poles, respectively.

Step 4. By Lemma 3.30 and Example 3.32, $\mathbb{R}^2$ with an abstract moment map Ψ is cobordant to $\mathbb{C}(a, \pm\infty)$ for the standard orientation and to $\overline{\mathbb{C}}(a, \pm\infty)$ otherwise, where $a = \Phi(0)$.

Step 5. A proper abstract moment map Ψ on the cylinder C factors through a proper function, also denoted by Ψ, on $\mathbb{R} = C/G$. The following three functions are of particular interest to us: $\Psi_+(t) = t$, $\Psi_-(t) = -t$, and $\Psi_0(t) = t^2$ where $t \in \mathbb{R}$.

Exercise. Prove that for any proper Ψ, the pair (C,Ψ) is cobordant to $(C,\Psi_\pm)$ or $(C,\pm\Psi_0)$.

Let C_+ denote the cobordism class of (C,Ψ_+). By Example 3.34, (C,Ψ) is either cobordant to zero or to $\pm C$. By Example 3.35, $C_+ = \mathbb{C}(a,-\infty)+\overline{\mathbb{C}}(a,+\infty)$.

This completes the proof of the theorem. □

Proof of Theorem 3.37. As is immediately clear from Theorem 3.36, the planes $\mathbb{C}(a,\pm\infty)$, $a\in\mathbb{R}$, generate Γ_{eff}. The relations (3.10) between $\mathbb{C}(a,\pm\infty)$ follow from (3.7) and (3.4). It remains to show that these are the only relations.

To this end, let us assume that

$$\sum_{i=1}^{m}\lambda_i^+\mathbb{C}(a_i,+\infty)+\lambda_i^-\mathbb{C}(a_i,-\infty)=0 \tag{3.11}$$

for some distinct $a_i\in\mathbb{R}$ and $\lambda_i^\pm\in\mathbb{Z}$, and show that (3.11) follows from (3.10).

Let (M,Ψ) be the disjoint union of appropriately oriented copies of $\mathbb{C}$ with quadratic moment maps as in Example 3.32 such that (M,Ψ) realizes the left-hand side of (3.11). For $x\in\mathbb{R}$ not equal to $a_1,\dots,a_m$, denote by N_x the algebraic number of points in $\Psi^{-1}(x)/G$ (counted with orientations). Since (M,Ψ) is cobordant to zero by (3.11), $\Psi^{-1}(x)/G$ is cobordant to zero as a zero-dimensional oriented manifold. Thus $N_x=0$ for all x. On the other hand,

$$N_{a_i+\epsilon}-N_{a_i-\epsilon}=\lambda_i^+-\lambda_i^-,$$

when $\epsilon>0$ is sufficiently small. Since the left-hand side is zero, we conclude that

$$\lambda_i^+=\lambda_i^-.$$

Set $\lambda_i=\lambda_i^+$, so that (3.11) turns into the equation

$$\sum_{i=1}^{m}\lambda_i\big(\mathbb{C}(a_i,+\infty)-\mathbb{C}(a_i,-\infty)\big)=0. \tag{3.12}$$

Let $x>\max\{a_1,\dots,a_m\}$. Then

$$\sum_i\lambda_i=\sum_i\lambda_i^+=N_x=0. \tag{3.13}$$

Fix $b\in\mathbb{R}$. As follows from (3.9),

$$\mathbb{C}(a_i,+\infty)-\mathbb{C}(a_i,-\infty)=\mathbb{C}(b,+\infty)-\mathbb{C}(b,-\infty).$$

By multiplying each of these equations by λ_i, adding them up, and taking into account (3.13), we obtain (3.12) or equivalently (3.11). Thus (3.11) does follow from (3.9). □

Let us now describe the full group Γ of cobordism classes of proper abstract moment maps on tame oriented G-surfaces. Denote by Γ_m, $m>0$, the group of such cobordism classes with G-actions which factor through the natural projection $G=S^1\to S^1/\mathbb{Z}_m$ and an $(S^1/\mathbb{Z}_m)$-action effective on every connected component. The following result is nearly obvious.

Theorem 3.39. *The identification of $G=S^1$ with $S^1=S^1/\mathbb{Z}_m$ gives rise to an isomorphism $\Gamma_m\to\Gamma_{\mathrm{eff}}$. The full group of cobordism classes Γ is isomorphic to*

$$\bigoplus_{m\in\mathbb{N}} \Gamma_m = \Gamma_{\text{eff}}^{\oplus\mathbb{N}}, \tag{3.14}$$

where $\mathbb{N}$ *denotes the set of positive integers.*

EXERCISE. Prove Theorem 3.39. Notice that (3.14) is based on two facts:

(1) There are no relations between groups Γ_m for different m's.
(2) The group of cobordism classes with trivial G-actions is zero.

The latter readily follows from the observation that the domain of a proper abstract map for a trivial G-action must be compact.

Finally, let us consider surfaces (M, Ψ) which are not necessarily tame. This means that we allow infinitely many connected components. Consider linear combinations

$$\sum_{i\in I} m_i \mathbb{C}(a_i, +\infty) + \sum_{j\in J} n_j \mathbb{C}(b_j, -\infty) \tag{3.15}$$

where $m_i, n_j \in \mathbb{Z}$ and $a_i, b_j \in \mathbb{R}$. For the moment map to be proper, we require that

- either I is finite or $a_i \to \infty$;
- either J is finite or $b_j \to \infty$;
- the moment maps are standard, that is, the moment map on the cylinder C_+ is $\Psi_+(e^{i\theta}, t) = t$, and the moment map on $\mathbb{C}$ are quadratic.

Moreover, we may assume that a_i, $i \in I$, are all different and that b_j, $j \in J$, are all different. With these assumptions, we have

THEOREM 3.40. *Every oriented surface with a locally effective circle action and proper abstract moment map is properly cobordant to a space of the form* (3.15).

Moreover, two surfaces of the form (3.15) are cobordant if and only if their difference is a finite combination of the relations (3.10). Finally, in allowing non-effective actions, there is a copy of the above group for each quotient $S^1/\mathbb{Z}_m$, $m \in \mathbb{N}$.

6. Cobordisms of linear actions

In Chapter 3 we will prove that (polarized) cobordism classes are generated by linear actions on vector spaces and on vector bundles. We will now show that there are no non-trivial relations between polarized cobordism classes of linear actions on vector spaces. (Note that cobordisms on the right- or left-hand side of (3.10) are not simultaneously polarized.)

Let G be a torus. Fix an element η of the Lie algebra $\mathfrak{g}$. Let $\Gamma_{\text{lin}}^{\eta}$ denote the subgroup of the cobordism group generated by η-polarized cobordism classes of oriented vector spaces V, equipped with linear G-actions and η-polarized abstract moment maps Ψ.

Let $[V, \alpha]$ denote the cobordism class of (V, Ψ) when $\Psi(0) = \alpha$. (In Proposition 4.18 we will show that any linear G action with $V^\eta = \{0\}$ admits an η-polarized moment map.)

PROPOSITION 3.41. *The group* $\Gamma_{\text{lin}}^{\eta}$ *is freely generated by the classes* $[V, \alpha]$*, for all* $\alpha \in \mathfrak{g}^*$ *and all isomorphism classes of linear* G*-actions with* $V^\eta = \{0\}$*. In other words, there no relations among these classes in* $\Gamma_{\text{lin}}^{\eta}$*.*

PROOF. Take any η-proper cobordism between $[V_i, \alpha_i]$, $i = 1, \ldots, N$. The fixed point set in the cobording manifold is a union of compact one-dimensional manifolds with boundary, whose boundary points are the origins of the V_i's, and possibly some additional compact manifolds without boundary that are irrelevant for this argument. Because a compact one-manifold with boundary is an interval, the vector spaces V_i are arranged in pairs. Since the cobording abstract moment map is constant on the fixed point set, $\alpha_i = \alpha_j$ for each such pair. For each component of the cobording fixed point set, the G-action on normal bundle is the same on all fibers, therefore, the representations V_i and V_j are isomorphic for each pair. □