CHAPTER 8

Quantization commutes with reduction

1. Quantization and reduction commute

Our goal in this chapter is to show that "quantization commutes with reduction". For the sake of simplicity we assume that G is a torus. Let (M, ω) be a compact symplectic manifold, equipped with a G-action and a moment map $\Phi\colon M \to \mathfrak{g}^*$. Let α be a regular value of Φ. The reduced space

$$M_\alpha = \Phi^{-1}(\alpha)/G$$

is a symplectic orbifold, with a symplectic form ω_α induced from ω. If the Hamiltonian G-manifold (M, ω, Φ) is quantizable and α is integral, the reduced space M_α is quantizable too: pre-quantization data on M gives rise to pre-quantization data on M_α. Thus, both the equivariant quantization of M and the ordinary quantization of M_α are well defined as indices of suitable Dolbeault operators. The quantization of M is a virtual representation of G and is determined by the multiplicities with which each weight $\beta \in \mathbb{Z}^*_G$ occurs in it. The quantization of M_α is a virtual vector space determined by its dimension. The "quantization commutes with reduction" theorem, often abbreviated as [Q,R]=0, asserts that the multiplicity with which the weight α occurs in the quantization of M is equal to the dimension of the quantization of the reduced space M_α:

$$\mathcal{Q}(M)^\alpha = \mathcal{Q}(M_\alpha). \tag{8.1}$$

We refer the reader to Chapter 6 for details on the notion of quantization that we are using.

The "quantization commutes with reduction" theorem was conjectured explicitly for the first time in [**GS3**]. However, less precise forms of it occur much earlier in the mathematics and physics literature; see, for instance, [**KKS**]. In the physics literature, this statement is as a special case of a general "meta-principle" which asserts that if one quantizes a classical physical system and then sets certain quantum mechanical observables equal to constants, one obtains the same "reduced" quantum mechanical system as that which one would obtain by setting the corresponding classical observables equal to constants and then quantizing this "reduced" classical system.

"Quantization commutes with reduction" has implications in representation theory, via the orbit method. For instance, it implies an old conjecture of A. Kirillov: Consider a representation ρ of G corresponding to a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ and a representation ρ' of $K \subset G$ corresponding to a coadjoint orbit $\mathcal{O}' \subset \mathfrak{k}^*$. Then if $\mathcal{O}'$ does not occur in the image of the projection $\mathcal{O} \to \mathfrak{k}^*$, the representation ρ' is not a sub-representation of $\rho|_K$.

We will give a cobordism proof of the "quantization commutes with reduction" theorem. In principle, this proof should require no assumptions on the fixed point set M^G, but, to make the idea of the proof as transparent as possible, we will assume that M^G is finite. The idea of the proof is the following: by the linearization theorem, the manifold is cobordant to the disjoint union of the linear tangent spaces at the fixed points:

$$M \sim \bigsqcup T_p M, \qquad p \in M^G. \tag{8.2}$$

Since cobordism commutes with reduction, the corresponding reduced spaces are cobordant:

$$M_\alpha \sim \bigsqcup (T_p M)_\alpha, \qquad p \in M^G. \tag{8.3}$$

Cobordant spaces have the same quantization. Therefore, to prove that "quantization commutes with reduction" for M, it suffices to prove this for the linear G-actions on T_pM.

Unfortunately, certain complications arise in this argument. The value α might not be regular for the moment map $T_pM \to \mathfrak{g}^*$. One can overcome this problem by working with nearby regular reduced spaces, which provide a desingularization of the singular reduced space. Then, another difficulty arises: the linear spaces T_pM come equipped with complex structures that are incompatible with their symplectic structure, and with such structures, "quantization commutes with reduction" is not always true at singular levels. In this chapter we tackle these problems and prove the following version of the "quantization commutes with reduction" theorem:

THEOREM 8.1 (Quantization commutes with reduction at regular values). *Let G be a torus and let (M, ω, Φ, J) be a quantizable stable complex Hamiltonian G-manifold. Suppose that M is compact and M^G is finite. Let $\alpha \in \mathbb{Z}_G^*$ be a weight for G which is a regular value for Φ. Then $\mathcal{Q}(M, \omega, \Phi, J)^\alpha = \mathcal{Q}(M_\alpha, \omega_\alpha, J_\alpha)$.*

Usually, one takes ω to be a symplectic structure and J a compatible almost complex structure, however, these assumptions are not necessary. "Quantization commutes with reduction" provides yet another example of a symplectic phenomenon which turns out to be of a topological nature.

When considering a *singular* value α of Φ, we do assume that ω is symplectic and J is a compatible almost complex structure (see Example D.12). Following [**MeSj**], we desingularize the reduced space M_α by considering a nearby non-empty regular reduced space $M_{\alpha+h} = Z/G$, for $Z = \Phi^{-1}(\alpha + h)$, where "nearby" means that $\alpha + h$ belongs to an alcove (a connected component of $\Phi(M)_{\text{reg}}$) whose closure contains α. On $M_{\alpha+h}$, we take the reduced stable complex structure $J_{\alpha+h}$, induced from J and work with a two-form ω_α defined as if reduction was carried out over the value α, not $\alpha + h$. The precise construction is described in Section 3 of Chapter 6. The quantization of $(M_\alpha, \omega_\alpha)$ is defined to be the Dolbeault index associated with this data. We denote it by

$$\mathcal{Q}(M_\alpha)_{+h} = \mathcal{Q}(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h}). \tag{8.4}$$

"Quantization commutes with reduction" then asserts that

$$\mathcal{Q}(M)^\alpha = \mathcal{Q}(M_\alpha)_{+h}. \tag{8.5}$$

The fact that the right-hand side is well defined (independent of h) was shown by Meinrenken and Sjamaar, [**MeSj**]. This also follows from the assertion (8.5).

When α is a regular value for Φ, we can still carry out the above "desingularization" procedure. Thus, we obtain

$$\mathcal{Q}(M_\alpha)_{+h} = \mathcal{Q}(M_\alpha), \tag{8.6}$$

because the right-hand side of (8.4) does not change as $\alpha + h$ varies through regular values of Φ. Thus, we recover (8.1) as a special case of (8.5).

REMARK 8.2. The reduced space $M_\alpha = \Phi^{-1}(\alpha)/G$ can be smooth (or an orbifold) even if α is singular. This happens when α is *quasi-regular*, meaning that all the points in $\Phi^{-1}(\alpha)$ have the same stabilizer (up to conjugacy, if G is non-abelian). In this case, we again obtain (8.6), see [**MeSj**]. This should also be provable by cobordism methods.

To summarize, let us state the theorem precisely:

THEOREM 8.3 ("Singular quantization commutes with reduction"). *Let a torus G act on a symplectic manifold (M, ω) with a moment map $\Phi \colon M \to \mathfrak{g}^*$. Suppose that (M, ω, Φ) is quantizable. Let J be a compatible almost complex structure. Suppose that M is compact and M^G is discrete. Let $\alpha \in Z_G^*$ be a weight for G.*

Suppose $\alpha \notin \Phi(M)$, so that $M_\alpha = \emptyset$. Then $\mathcal{Q}(M, \omega, \Phi, J)^\alpha = 0$.

Suppose that $\alpha \in \Phi(M)$. Let $\alpha + h$ belong to a component of $\Phi(M)_{\text{reg}}$ whose closure contains α. Then $\mathcal{Q}(M, \omega, \Phi, J)^\alpha = \mathcal{Q}(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h})$.

Note that here we must assume that ω is symplectic and J is compatible with ω. Otherwise, the moment map image $\Phi(M)$ is not a meaningful invariant. In Section 9 we give variants of Theorem 8.3 which do *not* assume non-degeneracy.

2. Quantization of stable complex toric varieties

In this section we state several propositions that can be viewed as "quantization commutes with reduction" theorems for linear spaces. The reduction of a linear space is a toric variety. However, if a linear space is equipped with incompatible symplectic and complex structures, its reduction is a *stable complex* toric variety. Thus, our first goal is to find the quantization of stable complex toric varieties.

Let us begin with the standard Kähler structure on $\mathbb{C}^d$. Let a torus G act on $\mathbb{C}^d$ with weights $-\alpha_1, \dots, -\alpha_d \in \mathbb{Z}_G^*$ and with a moment map $\Phi(z) = \frac{1}{2} \sum |z_j|^2 \alpha_j$. Assume that the weights are polarized (see Definition 3.19). The quantization of $\mathbb{C}^d$ with this structure was determined in Section 3 of Chapter 7: the multiplicity of each weight $\alpha \in \mathbb{Z}_G^*$ is given by

$$\left(\mathcal{Q}(\mathbb{C}^d)\right)^\alpha = N(\alpha),$$

where $N(\cdot)$ is the partition function associated with the weights α_j:

$$N(\alpha) = \# \left\{ m \in \mathbb{Z}_+^d \;\middle|\; \sum m_j \alpha_j = \alpha \right\}.$$

The reduced space is $(\mathbb{C}^d)_\alpha = \Phi^{-1}(\alpha)/G$. Then the "quantization commutes with reduction" theorem takes the following form:

PROPOSITION 8.4. *Let α is a regular value of Φ. Then* $\dim \mathcal{Q}((\mathbb{C}^d)_\alpha) = N(\alpha)$.

The reduced space $(\mathbb{C}^d)_\alpha$ is the Kähler toric variety corresponding to the polytope

$$\Delta = \{ s \in \mathbb{R}_+^d \;\mid\; \sum s_j \alpha_j = \alpha \}$$

(see Section 5 of Chapter 5) and $N(\alpha)$ is the number of lattice points in this polytope. Hence, Proposition 8.4 asserts:

> The quantization (Riemann–Roch number) of a toric variety corresponding to a polytope Δ is equal to the number of integral lattice points in Δ.

This assertion is a well-known "folk theorem" in the toric variety literature and is usually attributed to Atiyah or Danilov (see [**Dan**]) who worked with an algebro-geometric construction of toric varieties. We prove Proposition 8.4 in Section 6 of this chapter "by hand", following ideas of Sue Tolman (see [**KT1**]).

Consider now $\mathbb{C}^d$ equipped with a symplectic structure and a complex structure that are possibly incompatible. Explicitly, we take the standard complex structure J and a non-standard symplectic form

$$\omega^{\#} = \sum \epsilon_j dx_j \wedge dy_j,$$

where $\epsilon_1 = \ldots = \epsilon_r = -1$ and $\epsilon_{r+1} = \ldots = \epsilon_d = 1$. We let the torus G act, as before, with complex weights $-\alpha_1, \ldots, -\alpha_d$. The symplectic weights for this action are $-\alpha_j^{\#} = -\epsilon_j \alpha_j$; we assume that these weights are polarized. Let the a moment map be

$$\Phi^{\#}(z) = \nu + \frac{1}{2} \sum |z_j|^2 \alpha_j^{\#} \tag{8.7}$$

with $\nu \in \mathbb{Z}_G^*$. The quantization of this space was also calculated in Section 3 of Chapter 7: the multiplicity of each weight $\alpha \in \mathbb{Z}_G^*$ is given by

$$\left(\mathcal{Q}(\mathbb{C}^d, \omega^{\#}, \Phi^{\#}, J)\right)^{\alpha} = (-1)^r N(\alpha - \delta - \nu),$$

where $N(\cdot)$ is the partition function associated with the polarized weights $\alpha_j^{\#}$, for $j = 1, \ldots, d$, and

$$\delta = \sum_{j=1}^{r} \alpha_j^{\#}. \tag{8.8}$$

In this case, the "quantization commutes with reduction" principle takes the following form:

PROPOSITION 8.5. *Suppose that α is a regular value of the moment map $\Phi^{\#}$. Then the quantization of the reduced space is given by*

$$\dim \mathcal{Q}\left((\mathbb{C}^d)_{\alpha}^{\#}, \omega_{\alpha}^{\#}, J_{\alpha}\right) = (-1)^r N(\alpha - \delta - \nu).$$

Consider now a stable complex Hamiltonian G-manifold, (M, ω, Φ, J), to which we can apply the linearization theorem. Thus,

$$M \sim \bigsqcup T_p M;$$

see Chapter 4. Suppose that α is a regular value for the moment maps $M \to \mathfrak{g}^*$ and $T_p M \to \mathfrak{g}^*$ for all $p \in M^G$. From [Q,R]=0 for the linear spaces $T_p M$, one can deduce that [Q,R]=0 for M. We will give the precise argument in Section 3. However, there may exist values α which are regular for M but singular for some $T_p M$. Therefore, we must also consider singular reduction of linear spaces, even if our ultimate goal is to establish that quantization commutes with reduction for a regular value on M.

Suppose now that α is a singular value for $\Phi^{\#} \colon \mathbb{C}^d \to \mathfrak{g}^*$. We desingularize the reduced space as described in Section 3 of Chapter 6, by passing to a nearby regular value $\alpha+h$. "Nearby" means that $\alpha+h$ belongs to a connected component of $\Phi(M)_{\mathrm{reg}}$ whose closure contains α. "Quantization commutes with reduction" asserts that the resulting quantization has dimension $(-1)^r N(\alpha-\delta-\nu)$, where δ and ν are given in (8.7) and (8.8). This assertion is true if we make an additional assumption:

PROPOSITION 8.6. *If h is a positive linear combination of the non-polarized weights $\alpha_1,\dots,\alpha_d$, then*

$$\dim \mathcal{Q}((\mathbb{C}^d)^{\#}_{\alpha+h}, \omega^{\#}_{\alpha}, J_{\alpha+h}) = (-1)^r N(\alpha-\delta-\nu). \tag{8.9}$$

The proof of Proposition 8.6 will occupy Sections 4–8 of this chapter. Here we prove a special case:

PROOF OF A SPECIAL CASE OF PROPOSITION 8.6. Suppose that α is not in the "polarized moment cone"

$$C^{\#} = \nu + \sum \mathbb{R}_+ \alpha_j^{\#} = \mathrm{image}\Phi^{\#}.$$

Then the reduced space

$$(\mathbb{C}^d)^{\#}_{\alpha} = (\Phi^{\#})^{-1}(\alpha)/G$$

is empty. The right-hand side of (8.9) is then zero: otherwise, if there exist $m_j \geq 0$ such that $\alpha-\nu-\delta = \sum m_j\alpha_j^{\#}$, then

$$\alpha = \nu + \sum_{j=1}^{r}(m_j+1)\alpha_j^{\#} + \sum_{j=r+1}^{d} m_j\alpha_j^{\#}$$

is in $C^{\#}$, contradicting our assumption. Equation (8.9) then reads $0=0$. □

In the rest of this section and in the next section we prove that quantization commutes with reduction, assuming Proposition 8.6. Let us first prove Proposition 8.5:

PROOF OF PROPOSITION 8.5. When α is regular, the reduced space at α, $((\mathbb{C}^d)^{\#}_{\alpha}, \omega^{\#}_{\alpha}, J_{\alpha})$, is isomorphic to the nearby reduced space $((\mathbb{C}^d)^{\#}_{\alpha+h}, \omega^{\#}_{\alpha}, J_{\alpha+h})$. Proposition 8.5 then follows from Proposition 8.6. □

We will need the following two consequences of Proposition 8.6.

PROPOSITION 8.7. *Let $\alpha+h$ is in the "non-polarized moment cone"*

$$C = \nu + \sum_{j=1}^{d} \mathbb{R}_+ \alpha_j.$$

Then $\dim \mathcal{Q}((\mathbb{C}^d)^{\#}_{\alpha+h}, \omega^{\#}_{\alpha}, J_{\alpha+h}) = (-1)^r N(\alpha-\delta-\nu)$.

PROOF. By Proposition 8.6, it is sufficient to find a positive linear combination h' of α_j's such that $\alpha+h$ and $\alpha+h'$ belong to the same connected alcove (component of the set of regular values of $\Phi^{\#}$ in its image).

By assumption, there exist coefficients $s_j > 0$ such that $\alpha+h = \nu + \sum s_j\alpha_j$. Because the alcove containing $\alpha+h$ is an open convex polyhedral cone with vertex ν, it contains the entire ray $\nu + t\sum s_j\alpha$, $t>0$. For any α and a large enough t,

$$\alpha + t\sum s_j\alpha_j$$

will also be contained in this cone. Then $h' := \sum ts_j\alpha_j$ is a positive linear combination of α_j's such that $\alpha + h'$ is in the same alcove as $\alpha + h$. $\square$

We end the section with two examples which illustrate that we cannot remove the assumption on h in Proposition 8.6: [Q,R]=0 is not always true for reductions of $\mathbb{C}^d$ with incompatible symplectic and complex structures, if we desingularize by h which is not a positive linear combination of complex weights.

Example 8.8. Consider $\mathbb{C}$ with its standard complex structure and orientation, non-standard symplectic form $\omega^\# = -dx \wedge dy$, and circle action with moment map $\Phi^\#(z) = -\frac{1}{2}|z|^2$. Then $r = 1$, $\nu = 0$, $\delta = \alpha_j^\# = -1$, and the multiplicity of $\alpha \in \mathbb{Z}$ in the quantization of this space is equal to

$$(-1)^r N(\alpha - \delta) = \begin{cases} 0 & \alpha \geq 0, \\ -1 & \alpha \leq -1. \end{cases}$$

However, the quantization of the reduced space at $\alpha = 0$ is a virtual vector space of dimension -1, and it remains so if we "desingularize" by passing to $\alpha + h = h < 0$. Hence, for $\alpha = 0$ and $h < 0$, "[Q,R]=0" reads $0 = 1$.

The next example is even more interesting; it shows that, for singular values, [Q,R]=0 might not hold even if α is in the *interior* of the moment map image. Because an appropriate choice of h does give "[Q,R]=0" (by Proposition 8.6), we also see from this example that singular quantization might depend on the choice of desingularization if the symplectic and complex structures are incompatible.

Example 8.9. Consider $\mathbb{C}^3$ with the standard complex structure J, standard orientation, and the non-standard symplectic structure

$$\omega^\# = -dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3.$$

Let $S^1 \times S^1$ act on $(\mathbb{C}^3, \omega^\#)$ with a and moment map

$$\Phi^\#(z) = \frac{1}{2}\left(|z_1|^2 + |z_3|^2, |z_2|^2 + |z_3|^2\right).$$

Then $\nu = 0$, $\delta = \alpha_1^\# = (1,0)$, $\alpha_2^\# = (0,1)$, and $\alpha_3^\# = (1,1)$. Let $\alpha = (k,k)$ for some positive integer k. As follows from Section 1 of Chapter 7,

$$\begin{aligned} \left(\mathcal{Q}\left(\mathbb{C}^3, \omega^\#, \Phi^\#, J\right)\right)^\alpha &= (-1)^r N(\alpha - \delta - \nu) \\ &= -N(k-1, k) \\ &= -\#\{m \in \mathbb{Z}_+^3 \ \mid \ m_1 + m_3 = k-1 \text{ and } m_2 + m_3 = k\} \\ &= -k. \end{aligned}$$

On the other hand, if we desingularize by $h = (\epsilon_1, \epsilon_2)$ with $\epsilon_1 \neq \epsilon_2$, the quantization of the reduced space is, as is clear from Example 8.12 below,

$$\mathcal{Q}\left((\mathbb{C}^3)^\#_{\alpha+h}, \omega^\#_\alpha, J_{\alpha+h}\right) = \begin{cases} -(k+1) & \text{if } \epsilon_1 > \epsilon_2, \\ -k & \text{if } \epsilon_1 < \epsilon_2. \end{cases}$$

3. Linearization of [Q,R]=0

In this section we deduce that [Q,R]=0 for manifolds (see Section 1) from the [Q,R]=0 theorems for linear spaces, stated in Section 2.

Let (M, ω, Φ, J) be a Hamiltonian G-manifold and let $\mathbb{L} \to M$ be a pre-quantization line bundle. Suppose that M is compact and M^G is finite. For each fixed point $p \in M^G$, let $-\alpha_{1,p}, \dots, -\alpha_{d,p}$ be the isotropy weights at p with respect to J. Choose a polarizing vector, and let $-\alpha^{\#}_{1,p}, \dots, -\alpha^{\#}_{d,p}$ be the corresponding polarized weights (see Proposition 4.18). We may assume that the weights are ordered so that

$$\alpha^{\#}_{j,p} = \begin{cases} -\alpha_{j,p} & 1 \le j \le r_p, \\ \alpha_{j,p} & r_p < j \le d, \end{cases}$$

for some $1 \le r_p \le d$. Let

$$\delta_p = \sum_{j=1}^{r_p} \alpha^{\#}_{j,p}.$$

The multiplicity of each weight $\alpha \in \mathbb{Z}^*_G$ in the quantization of M is given by

$$\mathcal{Q}(M, \omega, \Phi, J)^{\alpha} = \sum_p (-1)^{r_p} N_p(\alpha - \delta_p - \Phi(p)), \tag{8.10}$$

where $N_p(\cdot)$ is the partition function associated with $\alpha^{\#}_{1,p}, \dots, \alpha^{\#}_{d,p}$; see Chapter 7.

The Hamiltonian Linearization Theorem (Theorem 4.10) provides the following proper Hamiltonian cobordism:

$$(M, \omega, \Phi, J) \sim \bigsqcup_p \left(T_pM, \omega^{\#}_p, \Phi^{\#}_p, J_p\right). \tag{8.11}$$

By Theorem 4.12, this cobordism carries the pre-quantization line bundles:

$$(M, \mathbb{L}, J) \sim \bigsqcup_p \left(T_pM, \mathbb{L}_p, J_p\right). \tag{8.12}$$

Let us now use this fact to prove the easiest case of the "quantization commutes with reduction" theorem, namely the case where α is regular for Φ and all $\Phi^{\#}_p$.

Proof of a special case of Theorem 8.1. Let us assume that the weight α is regular not just for Φ, but also $\Phi^{\#}_p$ for all $p \in M^G$. In Section 6 of Chapter 5 we showed that reduction and cobordism commute. Hence, by (8.11),

$$(M_\alpha, \omega_\alpha, J_\alpha) \sim \bigsqcup_p \left((T_pM)^{\#}_\alpha, (\omega^{\#}_p)_\alpha, (J_p)_\alpha\right). \tag{8.13}$$

Dolbeault quantization is invariant under compact cobordism. (For manifolds, cobordism invariance can be deduced indirectly from the Riemann–Roch theorem and Stokes' theorem. For orbifolds, this fact can be deduced from the Kawasaki Riemann–Roch theorem combined with Stokes' theorem; see Theorem I.19 of Appendix I. In Appendix J the cobordism invariance of the index is proved directly for manifolds and orbifolds.) Hence, (8.13) implies that

$$\dim \mathcal{Q}(M_\alpha, \omega_\alpha, J_\alpha) = \sum_p \dim \mathcal{Q}\left((T_pM)^{\#}_\alpha, (\omega^{\#}_p)_\alpha, (J_p)_\alpha\right). \tag{8.14}$$

By Proposition 8.5, which we will prove in Sections 4–8,

$$\dim \mathcal{Q}\left((T_pM)^{\#}_\alpha, (\omega^{\#}_p)_\alpha, (J_p)_\alpha\right) = (-1)^{r_p} N_p(\alpha - \delta_p - \Phi(p)). \tag{8.15}$$

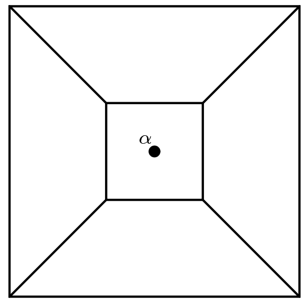

FIGURE 8.1. A value that is always singular for some T_pM

Equations (8.10), (8.14), and (8.15) give the desired equality

$$\mathcal{Q}(M,\omega,\Phi,J)^\alpha = \dim \mathcal{Q}(M_\alpha,\omega_\alpha,J_\alpha).$$

□

Unfortunately, there may exist values α which are regular for M but singular for some T_pM. Choosing a different "polarization" in the Linearization Theorem often allows one to bypass this problem. However, there may exist values α which are singular for all polarizations:

EXAMPLE 8.10. For $G = S^1 \times S^1$ there exists a six-dimensional Kähler Hamiltonian G-manifold M whose "X-ray" (i.e., roughly speaking, the moment map images of the orbit type strata) is given in Figure 8.1. (The manifold is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$, constructed as the fiberwise projectivization $M = \mathbb{P}(E \oplus \mathbb{C})$, where the holomorphic line bundle E is the tensor product $T\mathbb{CP}^1 \boxtimes T\mathbb{CP}^1$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the natural torus action.) Let α be the middle point of the X-ray. For any polarization, there exists a fixed point p such that α is singular for the moment map $\Phi_p^\# \colon T_pM \to \mathfrak{g}^*$.

We will treat the case that α is regular for M but singular for T_pM later in this section. Let us now assume that the closed two-form ω is symplectic and that J is an almost complex structure compatible with ω, and prove that quantization commutes with reduction for singular values.

PROOF OF THEOREM 8.3. The important consequence of the non-degeneracy of ω that we will use is the fact that the moment map image $\Phi(M)$ is contained in the "non-polarized moment cone" $C_p = \Phi(p) + \sum_j \mathbb{R}_+ \alpha_{j,p}$ for all p (see equation (2.14)):

$$\Phi(M) \subseteq C_p. \tag{8.16}$$

Consider a value $\alpha \in \mathbb{Z}_G^*$, which is possibly singular for Φ. Fix an alcove (connected component of $\Phi(M)_{\mathrm{reg}}$) whose closure contains α and choose $\alpha + h$ in this alcove so that $\alpha + th$ is a regular value for $\Phi_p^\# \colon T_pM \to \mathfrak{g}^*$ for all $p \in M^G$ and all $0 < t \leq 1$. The Linearization Theorem, followed by reduction with respect to the abstract moment map $\Phi_p^\# - h$, gives a cobordism

$$(M_{\alpha+h}, \mathbb{L}_\alpha, J_{\alpha+h}) \sim \bigsqcup \left((T_pM)^\#_{\alpha+h}, (\mathbb{L}_p)_\alpha, (J_p)_{\alpha+h} \right), \qquad p \in M^G,$$

where $\mathbb{L}$ is a pre-quantization line bundle for (M, ω, Φ). (Specifically, we use the items (1)–(3) of the Theorem 4.12 and the reduction cobordism described in Section 6 of Chapter 5.) This implies

$$\mathcal{Q}\left(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h}\right) = \sum \mathcal{Q}\left((T_pM)^{\#}_{\alpha+h}, (\omega^{\#}_p)_\alpha, (J_p)_{\alpha+h}\right), \qquad p \in M^G \tag{8.17}$$

by the cobordism invariance of the quantization (see Appendix J). By (8.16), $\alpha + h \in C_p$ for all p. By Proposition 8.7, (whose proof will be complete once we prove Proposition 8.6, in Sections 4–8),

$$\mathcal{Q}\left((T_pM)^{\#}_{\alpha+h}, (\omega^{\#}_p)_\alpha, (J_p)_{\alpha+h}\right) = (-1)^{r_p} N_p(\alpha - \delta_p - \Phi(p)).$$

By this, (8.17), and (8.10),

$$\left(\mathcal{Q}(M, \omega, \Phi, J)\right)^\alpha = \dim \mathcal{Q}\left(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h}\right).$$

□

We now return to the case where α is regular for $\Phi\colon M \to \mathfrak{g}^*$, but possibly singular for $\Phi^{\#}_p\colon T_pM \to \mathfrak{g}^*$, and ω is not necessarily symplectic. We could attempt to desingularize again by passing to $\alpha + h$. However, in general we cannot choose an h which is a positive linear combination of $\alpha_{1,p}, \dots, \alpha_{d,p}$ simultaneously for all p as we could do in the symplectic case of Theorem 8.3, thanks to the inclusion (8.16). Instead, we desingularize by deforming Φ in a slightly more sophisticated way.

Let $\mathbb{L}$ be a pre-quantization line bundle for (M, ω, Φ). Note that the reduced space of M depends on Φ and the quantization of M depends on $\mathbb{L}$ and Φ. We now separate their roles. Namely, in the proof below we replace Φ by an abstract moment map Φ' and we consider the new reduced spaces

$$M'_\alpha = (\Phi')^{-1}(\alpha)/G \tag{8.18}$$

with the pre-quantization line bundle

$$\mathbb{L}'_\alpha = (\mathbb{L}|_{\Phi^{-1}(\alpha)} \otimes \mathbb{C}_{-\alpha})/G. \tag{8.19}$$

Proof of Theorem 8.1. The *anti-canonical* line bundle of a complex manifold is the top wedge of its tangent bundle. Similarly, given a complex structure J on the vector bundle $E = TM \oplus \mathbb{R}^k$, the anti-canonical line bundle for the stable complex manifold (M, J) is defined to be $K = \bigwedge^m_{\mathbb{C}} E$ where $m = \operatorname{rank}_{\mathbb{C}} E$. Let Φ_K be the moment map corresponding to this line bundle. Explicitly, $\Phi^\xi_K = \Theta_K(\xi_P)$, where P is the unit circle bundle in K with respect to some G-invariant Hermitian metric and where Θ_K is a connection one-form on P. The value of Φ_K at each fixed point is the sum of the complex weights: $\Phi_K(p) = -\sum_{j=1}^d \alpha_{j,p}$.

Consider the abstract moment maps

$$\Phi + r\Phi_K, \qquad r \in \mathbb{R}.$$

The level sets of these maps are isomorphic to each other for r near zero. (This follows from the fact that the map $\mathbb{R} \times M \to \mathbb{R} \times \mathfrak{g}^*$ given by $(r, m) \mapsto (r, \Phi(m) + r\Phi_K(m))$ is proper, since M is compact, and $(0, \alpha)$ is a regular value.) We set

$$\Phi' := \Phi + r\Phi_K$$

where $r > 0$ is such a value. Because the α-level sets of Φ and Φ' are isomorphic, so are their quotients:

$$(M'_\alpha, \mathbb{L}'_\alpha, J'_\alpha) \cong (M_\alpha, \mathbb{L}_\alpha, J_\alpha),$$

where M'_α and $\mathbb{L}'_\alpha$ are given by (8.18) and (8.19). Therefore,

$$\mathcal{Q}(M_\alpha, \mathbb{L}_\alpha, J_\alpha) = \mathcal{Q}(M'_\alpha, \mathbb{L}'_\alpha, J'_\alpha). \tag{8.20}$$

Let us apply the linearization theorem (see Theorem 4.12) to the manifold M with the abstract moment map Φ', the line bundle $\mathbb{L}$, and the stable complex structure J. This gives a proper cobordism

$$(M, \Phi', \mathbb{L}) \sim \bigsqcup (T_pM, \Phi'_p, \mathbb{L}_p).$$

On the right-hand side, each moment map $\Phi'_p \colon T_pM \to \mathfrak{g}^*$ can be chosen arbitrarily as long as it is η-polarized and takes the value $\Phi'(p) = \Phi(p) - r(\alpha_{1,p} + \ldots + \alpha_{d,p})$ at the origin. Let us choose

$$\Phi'_p(\cdot) = \Phi^{\#}_p(\cdot) - r(\alpha_{1,p} + \ldots + \alpha_{d,p}). \tag{8.21}$$

If α is regular for Φ'_p for all p, then

$$\mathcal{Q}(M'_\alpha, \mathbb{L}'_\alpha, J'_\alpha) = \sum_{p \in M^G} \mathcal{Q}((T_pM)'_\alpha, (\mathbb{L}'_p)_\alpha, (J'_p)_\alpha). \tag{8.22}$$

This can be easily seen from the linearization cobordism (8.12), by applying reduction with respect to the abstract moment map Φ'. (See Chapter 5.)

By (8.21),

$$(T_pM)'_\alpha = (T_pM)^{\#}_{\alpha + h_p}, \qquad \text{where } h_p = r(\alpha_{1,p} + \ldots + \alpha_{d,p}).$$

Thus, *we have shifted the moment map by a different vector on each T_pM so that for each p the shift h_p is a positive linear combination of the non-polarized weights at p.* By Proposition 8.6,

$$\mathcal{Q}((T_pM)'_\alpha, (\mathbb{L}_p)'_\alpha, (J_p)'_\alpha) = (-1)^{r_p} N_p(\alpha - \delta_p - \Phi(p)) \tag{8.23}$$

where the prime denotes reduction with respect to Φ', when α is regular for $\Phi'|_{T_pM}$. Summing over all p's and applying (8.20), (8.22), and (8.10), we obtain that, if α is regular for all $\Phi'|_{T_pM}$,

$$\mathcal{Q}(M, \omega, J)^\alpha = \mathcal{Q}(M_\alpha, \omega_\alpha, J_\alpha),$$

as required.

However, α might be singular for $\Phi'|_{T_pM}$. This happens exactly when $\alpha + r(\alpha_{1,p} + \ldots + \alpha_{d,p})$ is singular for $\Phi^{\#}_p$. In this case, we desingularize further, by reducing Φ' at a value $\alpha + h'$ which is regular for all $\Phi'|_{T_pM}$ and such that h' is so small so that $r(\alpha_{1,p} + \ldots + \alpha_{d,p}) + h'$ is still a positive linear combination of $\alpha_{j,p}$'s for each p. By Proposition 8.6, the Φ'-reduction of T_pM at $\alpha + h$ has quantization of dimension $(-1)^{r_p} N_p(\alpha - \delta_p - \Phi(p))$. Arguing as before, we obtain

$$\begin{aligned}
\mathcal{Q}(M_\alpha, \mathbb{L}_\alpha, J_\alpha) &= \mathcal{Q}(M'_{\alpha+h'}, \mathbb{L}'_\alpha, J'_{\alpha+h'}) \\
&= \sum_{p \in M^G} \mathcal{Q}((T_pM)'_{\alpha+h'}, \mathbb{L}'_\alpha, J'_{\alpha+h'}) \\
&= \sum_{p \in M^G} N_p(\alpha - \delta_p - \Phi(p)) \\
&= \mathcal{Q}(M, \mathbb{L}, J)^\alpha,
\end{aligned}$$

as required. □

4. Straightening the symplectic and complex structures

We begin to prepare to the calculation of the Dolbeault quantization of a stable complex toric variety, as stated in Proposition 8.6. In this section we show how to replace the stable complex structure by a complex structure, and at what price.

First let us recall the setting. We consider $\mathbb{C}^d$ with the standard complex structure J, the action of the torus G act with weights $-\alpha_1, \dots, -\alpha_d$:

$$\exp(\xi)\colon (z_1, \dots, z_d) \mapsto (e^{-i\langle \alpha_1, \xi \rangle} z_1, \dots, e^{-i\langle \alpha_d, \xi \rangle} z_d) \quad \text{for all } \xi \in \mathfrak{g}, \tag{8.24}$$

and a non-standard symplectic form

$$\omega^\# = \sum \epsilon_j dx_j \wedge dy_j$$

with $\epsilon_1 = \ldots = \epsilon_r = -1$ and $\epsilon_{r+1} = \ldots = \epsilon_d = 1$ and a moment map $\Phi^\#$ such that $\Phi^\#(0) = \nu \in \mathbb{Z}_G^*$. We assume that the weights $\alpha_j^\# = \epsilon_j \alpha_j$ are polarized. The toric variety that we wish to quantize is the reduction of $(\mathbb{C}^d, \omega^\#, \Phi^\#, J)$ at α. We need to prove (8.9), i.e.,

$$\dim \mathcal{Q}((\mathbb{C}^d)^\#_{\alpha+h}, \omega^\#_\alpha, J_{\alpha+h}) \overset{\text{goal}}{=} (-1)^r N(\alpha - \delta - \nu), \tag{8.25}$$

where h is a positive linear combination of α_j's. The reduced space on the left-hand side stays the same if we replace the moment map $\Phi^\#$ by $\Phi^\# - \nu$ and the value α by $\alpha - \nu$. (This is the "shifting trick".) The right-hand side also stays the same (ν gets replaced by $\nu - \nu = 0$). Hence, without loss of generality, we may assume that the moment map vanishes at the origin:

$$\Phi^\#(z) = \frac{1}{2} \sum |z_j|^2 \alpha_j^\#.$$

The map

$$z_j \mapsto \begin{cases} z_j & j = 1, \dots, r, \\ \overline{z}_j & j = r+1, \dots, d \end{cases}$$

turns $\omega^\#$ into the standard symplectic structure ω, but transforms J to a non-standard complex structure. In other words, this map provides an isomorphism

$$(\mathbb{C}^d, \omega^\#, \Phi^\#, J) \cong (\mathbb{C}^d, \omega, \Phi^\#, J^\#),$$

where the G-action on the right-hand side is given by

$$\exp(\xi)\colon (z_1, \dots, z_d) \mapsto (e^{-i\langle \alpha_1^\#, \xi \rangle} z_1, \dots, e^{-i\langle \alpha_d^\#, \xi \rangle} z_d) \quad \text{for all } \xi \in \mathfrak{g} \tag{8.26}$$

and where ω is the standard symplectic form on $\mathbb{C}^d$ and $J^\#$ is a non-standard complex structure. (Note that $J^\#$ is equal to $-J$ on the first r coordinates.) Also, this transformation flips the orientation if r is odd. Hence, the toric variety that we wish to quantize can also be obtained as the reduction of of $(\mathbb{C}^d, \omega, \Phi^\#, J^\#)$ with respect to the G-action (8.26), with the orientation flipped if r is odd. In other words,

$$\left((\mathbb{C}^d)^\#_{\alpha+h}, \omega^\#_\alpha, J_{\alpha+h}\right) \cong (-1)^r \left((\mathbb{C}^d)^\#_{\alpha+h}, \omega_\alpha, J^\#_{\alpha+h}\right),$$

where the sign $(-1)^r$ indicates a possible orientation flip. Then equation (8.25) becomes

$$\dim \mathcal{Q}\left((\mathbb{C}^d)^\#_{\alpha+h}, \omega_\alpha, J^\#_{\alpha+h}\right) \overset{\text{goal}}{=} N(\alpha - \delta). \tag{8.27}$$

The pre-quantization line bundle for $(\mathbb{C}^d, \omega, \Phi^{\#}, J^{\#})$ is

$$\mathbb{L} = \mathbb{C}^d \times \mathbb{C},$$

where G acts on $\mathbb{C}^d$ as in (8.26) and acts on $\mathbb{C}$ trivially (because $\nu = 0$).

The quantization associated to a stable complex structure and a line bundle is completely determined by the corresponding Spin^c structure. By the "Spin^c shift formula", the G-equivariant Spin^c structure on $\mathbb{C}^d$ that is given by $J^{\#}$ and $\mathbb{L}$ is the same as the Spin^c structure that is associated with the standard complex structure J and the line bundle $\mathbb{L} \otimes \mathbb{C}_\delta$. See Section 3 of Appendix D; specifically, Example D.53.

Consider now the reduction of $\mathbb{C}^d$ with respect to some (abstract) moment map. If two equivariant stable complex structures and line bundles on $\mathbb{C}^d$ determine the same equivariant Spin^c structure, then, on the reduced space, the induced stable complex structures and line bundles also determine the same equivariant Spin^c structure, and hence the same quantization. (See Section 3.4 of Appendix D.) Therefore, equation (8.27) is equivalent to

$$\mathcal{Q}\left((\mathbb{C}^d)^{\#}_{\alpha+h}, \mathbb{L}_{\alpha-\delta}, J_{\text{red}}\right) \stackrel{\text{goal}}{=} N(\alpha - \delta). \tag{8.28}$$

Here, the quotient $(\mathbb{C}^d)^{\#}_{\alpha+h} = \left(\Phi^{\#}\right)^{-1}(\alpha+h)/G$ is taken with respect to the action (8.26), the stable complex structure J_{red} is induced from the standard complex structure on $\mathbb{C}^d$, and the pre-quantization line bundle is

$$\mathbb{L}_{\alpha-\delta} = (\mathbb{L} \otimes \mathbb{C}_\delta)_\alpha = \left((\mathbb{L}|_Z) \otimes \mathbb{C}_\delta \otimes \mathbb{C}_{-\alpha}\right)/G = Z \times_G \mathbb{C}_{\alpha-\delta},$$

where $Z = (\Phi^{\#})^{-1}(\alpha)$. The function $N(\cdot)$ is the partition function associated with $\alpha_1^{\#}, \dots, \alpha_d^{\#}$. We need to prove (8.28) when h is a positive linear combination of $\alpha_1, \dots, \alpha_d$.

5. Passing to holomorphic sheaf cohomology

Again, let us start by recalling the setting to which we have reduced the problem. We consider $\mathbb{C}^d$ with its standard complex and symplectic structures, and let the torus G act with weights $-\alpha_1^{\#}, \dots, -\alpha_d^{\#}$ that are polarized and with moment map $\Phi^{\#}(z) = \frac{1}{2}\sum |z_j|^2 \alpha_j^{\#}$. We have

$$\delta = \sum_{j=1}^{r} \alpha_j^{\#} \quad \text{and} \quad \alpha_j = \begin{cases} -\alpha_j^{\#} & j = 1, \dots, r, \\ \alpha_j^{\#} & j = r+1, \dots, d \end{cases}$$

for some $0 \le r \le d$. We consider a weight $\alpha \in \mathbb{Z}_G^*$, pick a nearby regular value $\alpha+h$, such that h is a positive linear combination of α_j's, and set $Z = (\Phi^{\#})^{-1}(\alpha+h)$. The toric variety that we wish to quantize is Z/G. This quotient is equipped with the stable complex structure J_{red} that is induced from the standard complex structure on $\mathbb{C}^d$, and the line bundle

$$\mathbb{L}_{\alpha-\delta} = Z \times_G \mathbb{C}_{\alpha-\delta}.$$

We need to show that the quantization of $(Z/G, \mathbb{L}_{\alpha-\delta}, J_{\text{red}})$ has dimension $N(\alpha-\delta)$.

The reduced space Z/G is actually a complex, not just stable complex, orbifold, as it can be identified with the G.I.T. quotient:

$$Z/G \cong W/G_{\mathbb{C}},$$

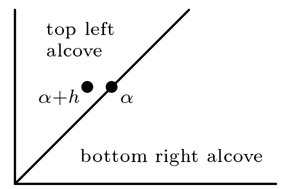

FIGURE 8.2. Moment map image for Example 8.12

where $W = G_{\mathbb{C}} \cdot Z$. (See Section 4 of Chapter 5.) Then, for any weight $\beta \in \mathbb{Z}_G^*$, the complex line bundle $\mathbb{L}_\beta = Z \times_G \mathbb{C}_\beta$ becomes the holomorphic line bundle

$$\mathbb{L}_\beta = W \times_{G_{\mathbb{C}}} \mathbb{C}_\beta.$$

The index of the corresponding Dolbeault operator is equal to the alternating sum

$$\sum_k (-1)^k \check{H}^k \left(W/G_{\mathbb{C}}, \mathcal{O}_{\mathbb{L}_\beta}\right)$$

of the cohomology groups of the sheaf of holomorphic sections; see Chapter 6. Setting $\beta = \alpha - \delta$, we get

$$\mathcal{Q}\left((\mathbb{C}^d)^{\#}_{\alpha+h}, \mathbb{L}_{\alpha-\delta}, J_{\text{red}}\right) = (-1)^r \sum_k (-1)^k \dim \check{H}^k \left(W/G_{\mathbb{C}}, \mathcal{O}_{\mathbb{L}_{\alpha-\delta}}\right). \tag{8.29}$$

To prove our goal, (8.28), it is enough to show that

$$\dim \check{H}^k \left(W/G_{\mathbb{C}}, \mathcal{O}_{\mathbb{L}_{\alpha-\delta}}\right) \overset{\text{goal}}{=} \begin{cases} N(\alpha-\delta) & k = 0, \\ 0 & k > 0. \end{cases} \tag{8.30}$$

REMARK 8.11. A formula for the sheaf cohomology of a holomorphic line bundle over a toric variety is given in [**Od**]. This formula implies (8.29) rather easily in the special case when $\delta = 0$ and α is a regular value.

We end this section with a holomorphic calculation that completes the proof of Example 8.9.

EXAMPLE 8.12. Let $G = S^1 \times S^1$ act on $\mathbb{C}^3$ by

$$(a, b) \cdot (z_1, z_2, z_3) = (a^{-1} z_1, b^{-1} z_2, (ab)^{-1} z_3) \tag{8.31}$$

with a moment map

$$\Phi^{\#}(z) = \frac{1}{2}\left(|z_1|^2 + |z_3|^2, |z_2|^2 + |z_3|^2\right).$$

The moment map image with its two alcoves is shown in Figure 8.2.

Let $\alpha = (k, k)$ for some positive integer k, and let $\delta = (1, 0)$. Recall that the space $W/G_{\mathbb{C}}$ depends on the choice of $\alpha + h$. We will show that if $\alpha + h$ is in the bottom right alcove,

$$\mathbb{L}_{\alpha-\delta} \to W/G_{\mathbb{C}} \tag{8.32}$$

is the line bundle $\mathcal{O}(k)$ over $\mathbb{CP}^1$, and the dimension of its quantization is $k+1$. If $\alpha + h$ is in the top left alcove, (8.32) is the line bundle $\mathcal{O}(k-1)$ over $\mathbb{CP}^1$, and the dimension of its quantization is k.

Let us prove these claims.

For $\alpha + h = (\kappa_1, \kappa_2)$ in either alcove, the level set $Z = (\Phi^{\#})^{-1}(\alpha + h)$ is given by the equations

$$\frac{1}{2}\left(|z_1|^2 + |z_3|^2\right) = \kappa_1 \quad \text{and} \quad \frac{1}{2}\left(|z_2|^2 + |z_3|^2\right) = \kappa_2. \tag{8.33}$$

On the bottom right alcove, $\kappa_1 > \kappa_2 > 0$. The equations (8.33) then imply that z_1 is never zero and that z_2 and z_3 cannot vanish simultaneously. By Theorem 5.18,

$$W = \{(z_1, z_2, z_3) \mid z_1 \neq 0 \text{ and } (z_2, z_3) \neq (0, 0)\}.$$

The quotient $W/G_{\mathbb{C}}$ is isomorphic to $\mathbb{CP}^1$ and an isomorphism is given by

$$[z_1, z_2, z_3] \mapsto [z_1 z_2, z_3], \tag{8.34}$$

where the left-hand side is an equivalence class in $W/G_{\mathbb{C}}$ for the $G_{\mathbb{C}}$ action (8.31).

Because $\alpha - \delta = (k-1, k)$, the line bundle $\mathbb{L}_{\alpha-\delta} \to W/G_{\mathbb{C}}$ can be identified with the set of equivalence classes

$$[z_1, z_2, z_3, u] \sim [az_1, bz_2, abz_3, a^{k-1}b^k u] \tag{8.35}$$

where $(z_1, z_2, z_3) \in W$, $u \in \mathbb{C}$, and $(a, b) \in (\mathbb{C}^\times)^2$. The map

$$[z_1, z_2, z_3, u] \mapsto [z_1 z_2, z_3, z_1 u]$$

sends it to the line bundle over $\mathbb{CP}^1$ with Chern class k, whose total space is given by the equivalence classes

$$[w_1, w_2, u] \sim [cw_1, cw_2, c^k u], \qquad w_1, w_2, u \in \mathbb{C}, \quad (w_1, w_2) \neq (0, 0).$$

For this line bundle, $\dim H^0 = k + 1$ and $\dim H^{>0} = 0$. By (8.29), the dimension of its quantization is $k + 1$.

On the top left alcove, $\kappa_2 > \kappa_1 > 0$. The equations (8.33) then imply that

$$W = \{(z_1, z_2, z_3) \mid z_2 \neq 0 \text{ and } (z_1, z_3) \neq (0, 0)\}.$$

As before, the quotient $W/G_{\mathbb{C}}$ is isomorphic to $\mathbb{CP}^1$ through the formula (8.34) and the line bundle $\mathbb{L}_{\alpha-\delta}$ is given by the formula (8.35). This line bundle, however, is isomorphic to the line bundle over $\mathbb{CP}^1$ with Chern class $k - 1$; an isomorphism is

$$[z_1, z_2, z_3, u] \mapsto [z_1 z_2, z_3, z_2^{-1} u].$$

The dimension of its quantization is k.

6. Computing global sections; the lit set

In this section we will compute the space of global holomorphic sections. More specifically, we will show that its dimension is

$$\dim \check{H}^0(W/G_{\mathbb{C}}, \mathcal{O}_{\mathbb{L}_{\alpha-\delta}}) = N(\alpha - \delta), \tag{8.36}$$

proving the first part of our goal (8.30).

It will be convenient to work with an "upstairs description" of $\mathcal{O}_{\mathbb{L}_{\alpha-\delta}}$, in terms of holomorphic functions on an open subset of $\mathbb{C}^d$: the holomorphic sections of the line bundle

$$\mathbb{L}_{\alpha-\delta} = W \times_{G_{\mathbb{C}}} \mathbb{C}_{\alpha-\delta} \to W/G_{\mathbb{C}}$$

can be viewed as G-equivariant holomorphic functions $W \to \mathbb{C}_{\alpha-\delta}$, i.e., holomorphic functions on W that have the transformation property

$$f((\exp \xi) \cdot z) = e^{i\langle \alpha-\delta, \xi\rangle} f(z) \quad \text{for all } \xi \in \mathfrak{g}. \tag{8.37}$$

Because W contains $(\mathbb{C}\smallsetminus\{0\})^d$, a holomorphic function on W can be expanded into a Laurent series

$$f = \sum_{m\in\mathbb{Z}^d} a_m z^m. \tag{8.38}$$

This function satisfies (8.37) if and only if

$$\sum m_j \alpha_j^{\#} = \alpha - \delta \tag{8.39}$$

for every $m = (m_1, \dots, m_d)$ such that $a_m \neq 0$.

Holomorphic monomials on $\mathbb{C}^d$ are precisely those z^m for which $m_j \geq 0$ for all j. On W, a priori, there could be holomorphic monomials with some $m_j < 0$. However, the following lemma says that this cannot happen:

LEMMA 8.13. *Suppose that $m \in \mathbb{Z}^d$ satisfies* (8.39) *and the monomial z^m is holomorphic on W. Then $m_j \geq 0$ for all j.*

Hence, the space of holomorphic sections is spanned by the monomials z^m such that $m \in \mathbb{Z}^d$ satisfies (8.39) and $m_j \geq 0$ for all j. The number of such monomials is precisely $N(\alpha - \delta)$, by the definition of the partition function $N(\cdot)$. This proves (8.36).

REMARK 8.14. If the complement of W in $\mathbb{CP}^d$ has complex codimension greater than or equal to 2, the holomorphic monomial z^m extends to all of $\mathbb{C}^d$, by Hartog's theorem (see, e.g., [**GH**]). Hence, $m_j \geq 0$ for all j. However, if the level set $\Phi^{-1}(\alpha)$ does not meet the coordinate plane $\{z_j = 0\}$, the complement of W contains this plane and, hence, has codimension one.

The remainder of this section is devoted to the proof of Lemma 8.13.

The monomial z^m is holomorphic on W if and only if, for each j, the exponent $m_j \geq 0$ once there exists $z \in W$ with $z_j = 0$. Because $W = G_{\mathbb{C}} \cdot Z$, it is enough to check this criterion for z in Z. Moreover, since $z \in Z$ is equivalent to $\frac{1}{2}\sum |z_j|^2 \alpha_j = \alpha + h$, there exists $z \in Z$ with $z_j = 0$ if and only if there is $s \in \Delta$ with $s_j = 0$, where

$$\Delta = \Delta_{\alpha+h} = \{s \in \mathbb{R}^d_+ \mid \sum s_j \alpha_j = \alpha + h\}.$$

For any $m \in \mathbb{R}^d$, let us define the corresponding *lit set* $L_m \subset \partial\Delta$ as

$$L_m = \{s \in \Delta_{\alpha+h} \;\mid\; \exists\, i \text{ such that } s_i = 0 \text{ and } m_i < 0\}. \tag{8.40}$$

Then for $m \in \mathbb{Z}^d$, the monomial z^m is holomorphic on W if and only if the lit set L_m is empty. We need to prove that for each $m \in \mathbb{Z}^d$ that satisfies (8.39), $m_j < 0$ for some j implies that the lit set is non-empty.

The lit set has an intuitively clear meaning, which justifies its name, in the case that m belongs to the affine space spanned by Δ,

$$A(\alpha + h) = \{x \in \mathbb{R}^d \mid \sum x_j \alpha_j = \alpha + h\}.$$

Such an m is outside Δ if and only if $m_j < 0$ for some j. Suppose that this is the case. Now, standing at m, let us point a flashlight at Δ. Because Δ is a solid polytope that does not let light through, the jth facet is lit if and only if light rays from m reach that facet from the outside, without having to pass through the interior of the polytope. This exactly means that $m_j < 0$. Therefore, the set of points that are lit by the flashlight is precisely L_m. This is the reason for the name "lit set".

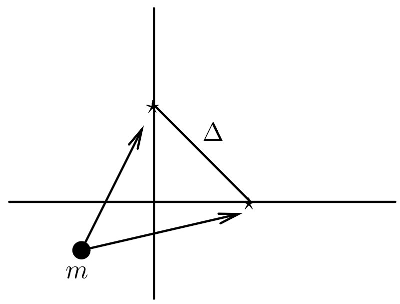

FIGURE 8.3. Lit set $\star$ when m is not coplanar with Δ

REMARK 8.15. The notion of a *visibility set* is sometimes used in Riemannian geometry: L_m is the set of points of Δ that are visible to a spectator standing at m.

If we stand outside a convex polytope and point a flashlight at the polytope, the lit set is non-empty and contractible. Therefore, if m belongs to $A(\alpha + h)$ but is not in Δ, the lit set L_m is non-empty and contractible.

For certain m's which are not in the affine space spanned by Δ, these properties are not always satisfied: the following example shows that, if m lies outside the affine space spanned by Δ, the lit set might not be contractible:

EXAMPLE 8.16. Take $L(x, y) = x + y$, $\alpha = 0$, $\alpha + h = \epsilon$, and $m = (m_1, m_2)$ with both m_1 and m_2 negative. Then Δ is an interval and L_m is its boundary, which is not contractible. See Figure 8.3.

Let us now restrict our attention to $m \in \mathbb{Z}^d$ such that $\sum m_j \alpha_j^\# = \alpha - \delta$. Such an m is not in the affine plane $A(\alpha + h)$. However, the lit set L_m is still non-empty and contractible.

PROPOSITION 8.17. *Suppose that h is a positive linear combination of α_j's and that $\delta = \sum_{j=1}^r \alpha_j^\# = -\sum_{j=1}^r \alpha_j$. Let $m \in \mathbb{Z}^d$ be such that $\sum m_i \alpha_i^\# = \alpha - \delta$. Suppose also that $m_j < 0$ for some j. Then the lit set L_m is non-empty and contractible.*

PROOF. Let

$$h = \sum h_j \alpha_j^\#$$

with $h_j > 0$ for all j. We may assume, by shrinking h if necessary, that $h_j < 1$ for all j. Let

$$m'_j = \begin{cases} m_j + 1 - h_j & 1 \le j \le r, \\ m_j + h_j & r < j \le d. \end{cases}$$

Because $m_j \in \mathbb{Z}$ and $0 < h_j < 1$, we have

$$m_j < 0 \iff m'_j < 0 \tag{8.41}$$

for each j. Also, because $\alpha_j^\# = -\alpha_j$ for $1 \le j \le r$ and $\alpha_j^\# = \alpha_j$ for $r < j \le d$, we have

$$\sum_{j=1}^d m'_j \alpha_j^\# = \sum_{j=1}^d m_j \alpha_j^\# + \sum_{j=1}^r \alpha_j^\# + \sum_{j=1}^d h_j \alpha_j = (\alpha - \delta) + \delta + h = \alpha + h.$$

Thus m' is in $A(\alpha+h)$. By (8.41) and the hypothesis, $m'_j < 0$ for some j. Hence, m' is outside the polytope $\Delta_{\alpha+h}$. As pointed out above, these properties of m' imply that the lit set $L_{m'}$ is non-empty and contractible. Because the lit set depends only on the collection of j's for which $m_j < 0$, it follows from (8.41) that $L_m = L_{m'}$. Hence, L_m is non-empty and contractible. □

Proof of Lemma 8.13. Suppose that the monomial z^m, for $m \in \mathbb{Z}^d$, is holomorphic on W. As we have seen, this means that the lit set L_m is empty. Suppose in addition that m satisfies (8.39). Then, by Proposition 8.17, $m_j \geq 0$ for all j. □

7. The Čech complex

It remains to prove the second part of (8.30), i.e.,

$$\dim \check{H}^{>0}\left(W/G_{\mathbb{C}}, \mathcal{O}_{\mathbb{L}_{\alpha-\delta}}\right) \overset{\text{goal}}{=} 0, \tag{8.42}$$

where $W = G_{\mathbb{C}} \cdot Z$ and $Z = \Phi^{-1}(\alpha + h)$, assuming that h is a positive linear combination of α_j's. We will calculate this sheaf cohomology from a "good cover" of $W/G_{\mathbb{C}}$. We do not need to look very far: in Section 5 of Chapter 5 we obtained the following explicit description of W:

$$W = \bigcup_I W_I, \quad I \in \mathcal{F}_{\alpha+h}, \tag{8.43}$$

where

$$W_I = \mathbb{C}^I \times (\mathbb{C}^\times)^{d \smallsetminus I},$$

and where $\mathcal{F}_{\alpha+h}$ denotes the collection of subsets $I \subseteq \{1, \ldots, d\}$ such that there exists $z \in (\Phi^{\#})^{-1}(\alpha + h)$ for which $z_i = 0$ exactly if $i \in I$.

We consider the resulting covering of the reduced space,

$$\overline{\mathbb{W}} = \{\overline{W}_I \mid I \in \mathcal{F}_{\alpha+h}\},$$

where $\overline{W}_I = W_I/G_{\mathbb{C}}$. Consider the holomorphic line bundle $\mathbb{L}_\beta = W \times_{G_{\mathbb{C}}} \mathbb{C}_\beta$ for any $\beta \in \mathbb{Z}^*_G$; we will soon specialize to $\beta = \alpha - \delta$.

Claim 8.18. *For each $I \in \mathcal{F}_{\alpha+h}$, the Dolbeault cohomology of $\overline{W}_I$ twisted by the line bundle $\mathbb{L}_\beta|_{\overline{W}_I}$ vanishes in all positive degrees.*

Proof. By the results of Section 5.4 of Chapter 5, each of the sets $W_I/G_{\mathbb{C}}$ is biholomorphically equivalent to a space of the form $\left(\mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}\right)/\Gamma$, where Γ is a finite abelian group acting linearly on $\mathbb{C}^n$. The restriction of the line bundle $\mathbb{L}_\beta$ to this open set is

$$\left(\mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}\right) \times_\Gamma \mathbb{C}$$

for some linear action of Γ on $\mathbb{C}$. It follows that the Dolbeault cohomology in question can be identified with Γ-invariant Dolbeault cohomology with coefficients in the trivial line bundle $\left(\mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}\right) \times \mathbb{C}$, where the Γ-action on the fiber is perhaps non-trivial. An averaging argument shows that this cohomology is the same as the Dolbeault cohomology of $\mathbb{C}^k \times (\mathbb{C}^\times)^{n-k}$ with trivial coefficients. By the $\overline{\partial}$-Poincaré lemma (see, e.g., [**GH**]), the latter is acyclic. □

Since $W_I \cap W_J = W_{I \cap J}$, the cover $\overline{\mathbb{W}}$ has the property that any intersection of sets in the cover is acyclic with respect to the Dolbeault cohomology, i.e., satisfies $H^{>0}(V, \mathcal{O}_{\mathbb{L}_\beta}) = 0$. By Leray's theorem (see, e.g., [**GH**]), this implies that the sheaf cohomology groups of $\mathcal{O}_{\mathbb{L}_\beta}$ over $W/G_{\mathbb{C}}$ are equal to the cohomology groups of the Čech complex for the cover $\overline{\mathbb{W}}$:

$$\check{C}\left(\overline{\mathbb{W}}, \mathcal{O}_{\mathbb{L}_\beta}\right).$$

A k-cochain c in this complex is a map which assigns to every $(k+1)$-tuple of multi-indices $I_0, \ldots, I_k$ in $\mathcal{F}_{\alpha+h}$ a holomorphic section of $\mathbb{L}_\beta$ over the intersection $\overline{W}_{I_0} \cap \ldots \cap \overline{W}_{I_k}$. This intersection is equal to $\overline{W}_I$ for $I = I_0 \cap \ldots \cap I_k$, and the line bundle over this intersection is

$$\mathbb{L}_\beta|_{\overline{W}_I} = W_I \times_{G_{\mathbb{C}}} \mathbb{C}_\beta.$$

Recall that $W_I = \mathbb{C}^I \times (\mathbb{C}^\times)^{d \smallsetminus I}$. Let us argue as in Section 6: holomorphic sections of $\mathbb{L}_\beta$ over W_I can be viewed as holomorphic functions on W_I whose Laurent expansion has the form

$$f = \sum a_m z^m,$$

where

$$m_j \geq 0 \text{ for all } j \in I \text{ and } \sum m_i \alpha_i^\# = \beta. \tag{8.44}$$

Denote by $\mathcal{O}_m$ the sheaf of functions on $\mathbb{C}^d$ which are multiples of the monomial z^m. One may think of the space $\mathcal{O}_{\mathbb{L}_\beta}(W_I)$ as the direct sum of the spaces $\mathcal{O}_m(W_I)$, over all $m \in \mathbb{Z}^d$ such that $\sum m_i \alpha_i^\# = \beta$. (Strictly speaking, this is only true for a completion of $\mathcal{O}_\beta(W_I)$ unless the direct sum is finite.) Motivated by this observation, we now concentrate on one particular value of m.

The sheaf $\mathcal{O}_m$ can be described in the following way. The complex torus $(\mathbb{C}^\times)^d$ acts on all ingredients involved: on $\mathbb{C}^d$, on $W/G_{\mathbb{C}}$, and on $\mathbb{L}_\beta$. These actions induce an action on the sheaf of holomorphic sections. The monomial $f(z) = z^m$ transforms as

$$f(\lambda z) = \lambda_1^{m_1} \ldots \lambda_d^{m_d} f(z), \qquad \lambda \in (\mathbb{C}^\times)^d. \tag{8.45}$$

Because the torus element $\lambda \in (\mathbb{C}^\times)^d$ acts on functions by sending a function $f(z)$ to the function $f(\lambda^{-1} z)$, the functions which transform according to (8.45) are those functions on which the torus acts with weight $-m$.

The $(\mathbb{C}^\times)^d$-action commutes with the Čech differential

$$(Dc)_{J_0, \ldots, J_{k+1}} = \sum_{l=0}^{k+1} (-1)^l c_{J_0, \ldots, \hat{J}_l, \ldots, J_{k+1}}. \tag{8.46}$$

Therefore, this action descends to the cohomology. Moreover, for each m, the cochains on which the torus acts with weight $-m$ form a subcomplex, and its cohomology is the subspace of the sheaf cohomology consisting of those classes on which the torus $(\mathbb{C}^\times)^d$ acts with weight $-m$.

Because $W/G_{\mathbb{C}}$ is compact, the cohomology is finite-dimensional. Therefore, we have a genuine decomposition

$$\check{H}(\overline{\mathbb{W}}, \mathcal{O}_{\mathbb{L}_\beta}) \cong \bigoplus_m \check{H}(\mathbb{W}, \mathcal{O}_m), \qquad m \in \mathbb{Z}^d, \qquad \sum m_i \alpha_i^\# = \beta \tag{8.47}$$

for the covering

$$\mathbb{W} = \{W_I \mid I \in \mathcal{F}_{\alpha+h}\}.$$

(In other words, all but a finite number of terms on the right of (8.47) vanish.) Therefore, we need to prove that

$$\check{H}^{>0}(\mathbb{W}, \mathcal{O}_m) \overset{\text{goal}}{=} 0 \quad \text{for all } m \in \mathbb{Z}^d \quad \text{such that } \sum m_i \alpha_i^{\#} = \alpha - \delta. \tag{8.48}$$

The k-cochains in the complex which gives $\check{H}(\mathbb{W}, \mathcal{O}_m)$ are

$$\bigoplus_{I_0, \dots, I_k \in \mathcal{F}_{\alpha+h}} \mathcal{O}_m(W_{I_0} \cap \ldots \cap W_{I_k}). \tag{8.49}$$

Recall that $W_{I_0} \cap \ldots \cap W_{I_k} = W_I$ for $I = I_0 \cap \ldots \cap I_k$. The following lemma follows immediately from the fact that $W_I = \mathbb{C}^I \times (\mathbb{C}^\times)^{d \smallsetminus I}$:

Lemma 8.19. *If $m_j \geq 0$ for all $j \in I$, the space $\mathcal{O}_m(W_I)$ is one-dimensional and spanned by $z_1^{m_1} \dots z_d^{m_d}$. Otherwise, $\mathcal{O}_m(W_I) = \{0\}$.*

Hence, the complex (8.49) giving $\check{H}(\mathbb{W}, \mathcal{O}_m)$ is isomorphic to the complex

$$\Omega_m^k = \bigoplus_{I_0, \dots, I_k \in \mathcal{F}_{\alpha+h}} \begin{cases} \mathbb{C} & \text{if } m_i \geq 0 \text{ for all } i \in I_0 \cap \ldots \cap I_k, \\ 0 & \text{otherwise.} \end{cases} \tag{8.50}$$

The differential D in $\Omega_m^\star$ comes from the standard formula for the Čech differential: for $c \in \Omega_m^{k-1}$,

$$(Dc)_{I_0, \dots, I_k} = \sum_{j=0}^{k} (-1)^j c_{I_0, \dots, \hat{I}_j, \dots, I_k}.$$

It remains to show that the higher cohomology $H^{>0}(\Omega_m^k, D)$ vanishes when $m \in \mathbb{Z}^d$ and $\sum m_i \alpha_i^{\#} = \alpha - \delta$.

8. The higher cohomology

Our goal now is to prove that

$$\check{H}^{>0}(\Omega_m^\star, D) = 0 \quad \text{for all } m \in \mathbb{Z}^d \quad \text{such that } \sum m_i \alpha_i^{\#} = \alpha - \delta, \tag{8.51}$$

where $\Omega_m^\star$ is the complex (8.50). We will show that the cohomology of this complex is essentially equal to the relative cohomology of the pair (Δ, L_m), where Δ is the polytope

$$\Delta = \Delta_{\alpha+h} = \{s \in \mathbb{R}_+^d \mid \sum s_i \alpha_i^{\#} = \alpha + h\}$$

and L_m is the "lit set"

$$L_m = \{s \in \Delta \mid \exists\, i \text{ such that } s_i = 0 \text{ and } m_i < 0\}.$$

Because both Δ and L_m are contractible, this cohomology will be zero.

Recall that the elements of $\mathcal{F}_{\alpha+h}$ encode the faces of the polytope Δ:

$$\mathcal{F}_{\alpha+h} = \mathcal{F}_\Delta = \{I \subseteq \{1, \dots, d\} \mid \exists\, s \in \Delta \text{ such that } s_i = 0 \iff i \in I\}.$$

The covering of W by the sets W_I, for $I \in \mathcal{F}_\Delta$, corresponds to the covering of Δ by the sets

$$\begin{aligned} U_I &= \{s \in \Delta_{\alpha+h} \mid s_i \neq 0 \text{ for all } i \notin I\} \\ &= \text{ the star of the } I\text{th face of } \Delta, \end{aligned} \tag{8.52}$$

for $I \in \mathcal{F}_\Delta$. Namely, W_I consists of exactly those elements of W which are sent to U_I by the $\mathbb{T}^d$-moment map

$$W \to W/G_\mathbb{C} \cong Z/G \to \Delta_\alpha,$$

the last arrow maps $[z_1, \dots, z_d]$ to $\frac{1}{2}(|z_1|^2, \dots, |z_d|^2)$. More explicitly, U_I is the union of the interiors of faces whose closures contain the Ith face of Δ. We consider the covering of Δ by the open sets U_I.

Each U_I is a non-empty contractible open set, and $U_{I\cap J} = U_I \cap U_J$ for all I and J. Therefore, the sets U_I form a *good covering* for Δ, in the usual sense: the Čech cohomology of this covering, with constant complex coefficients, is equal to the singular cohomology of Δ. See [**BT1**, Theorem 15.8]. Because the intersections $U_{I_0} \cap \dots \cap U_{I_k}$ are never empty, the Čech complex for this covering is

$$C^k = \bigoplus_{I_0,\dots,I_k \in \mathcal{F}_{\alpha+h}} \mathbb{C}. \tag{8.53}$$

The complex $C^\star$ coincides with the complex $\Omega^\star_m$ (see (8.50)) if $m_j \geq 0$ for all j. Hence, in this case, the cohomology groups of $\Omega^\star_m$ are identical to the cohomology groups of the (contractible) space Δ_α:

$$H^k(\Omega^\star_m) = H^k(C^\star) = H^k(\Delta) = \begin{cases} \mathbb{C} & k = 0, \\ 0 & k > 0. \end{cases}$$

Therefore, $H^{>0}(\Omega^\star_m) = 0$, if $m_j \geq 0$ for all j, as required.

It remains to prove that $H^{>0}(\Omega^\star_m) = 0$ for all m that satisfy

$$m \in \mathbb{Z}^d, \qquad \sum m_i \alpha_i^\# = \alpha - \delta, \qquad \text{and } \exists\ j \text{ such that } m_j < 0. \tag{8.54}$$

It will be convenient to express the complex $\Omega^\star_m$ as a quotient:

$$\Omega^k_m = C^k / Q^k_m,$$

where C^k is given by (8.53) and

$$Q^k_m = \bigoplus_{I_0,\dots,I_k \in \mathcal{F}} \begin{cases} 0 & \text{if } m_i \geq 0 \text{ for all } i \in I_0 \cap \dots \cap I_k, \\ \mathbb{C} & \text{otherwise .} \end{cases} \tag{8.55}$$

It is enough to show that

$$H^{>0}(Q^\star_m) \overset{\text{goal}}{=} 0 \tag{8.56}$$

for all m that satisfy (8.54). (The conclusion that $\check{H}^k(\Omega^\star_m) = 0$ for $k > 0$ follows then from the long exact sequence in cohomology induced by the short exact sequence of complexes $0 \to Q^\star_m \to C^\star \to \Omega^\star_m \to 0$.) Our proof will be complete once we show that

$$H^k(Q^\star_m) = H^k(L_m), \tag{8.57}$$

because the lit set L_m is contractible (by Proposition 8.17).

Consider the covering $\mathcal{V}_m$ of the lit set L_m by the sets

$$V_I = U_I \cap L_m, \qquad I \in \mathcal{F}_\Delta.$$

Claim 8.20. *The covering $\mathcal{V}_m$ is a good covering of the lit set L_m.*

Proof. Recall that U_I is the star of the Ith face of Δ. If the Ith face of Δ is not lit, the entire star is not lit, and V_I is empty. If the Ith face of Δ is lit, then V_I is equal to its star in L_m, which is contractible. Also, just as for the U_I's,

$$V_{I_0} \cap \dots \cap V_{I_k} = V_I \quad \text{for} \quad I = I_0 \cap \dots \cap I_k.$$

It follows that $\mathcal{V}_m$ is a good covering, as required. □

REMARK 8.21. $\mathcal{V}_m$ is an *indexed* covering of $\mathbb{L}_m$; the set V_I can be equal to the set V_J even if $I \neq J$. This does not cause any trouble; the Čech theory still goes through for indexed coverings.

CLAIM 8.22. *$V_I \neq \emptyset$ if and only if there exists $j \in I$ such that $m_j < 0$.*

PROOF. Suppose $j \in I$ and $m_j < 0$. Choose a point s in the jth open facet of Δ. Then $s \in U_I$ and s is lit.

Conversely, suppose that $s \in U_I$ is lit. The condition that $s \in U_I$ implies that $s_l > 0$ for all $l \notin I$. The condition that s be lit implies that there exists j such that $s_j = 0$ and $m_j < 0$. Combining these two observations, we conclude that j must be in I. □

CLAIM 8.23. *The Čech complex for the covering $\mathcal{V}_m$, with constant complex coefficients, coincides with the complex $Q_m^\star$.*

PROOF. The Čech complex is

$$\check{C}^k(\mathcal{V}_m;\mathbb{C}) = \bigoplus_{I_0,\dots,I_k \in \mathcal{F}_{\alpha+h}} \begin{cases} \mathbb{C} & \text{if } V_{I_0} \cap \dots \cap V_{I_k} \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The complex $Q_m^\star$ (see (8.55)) can be rewritten as

$$Q_m^k = \bigoplus_{I_0,\dots,I_k \in \mathcal{F}_{\alpha+h}} \begin{cases} \mathbb{C} & \text{if } \exists\, i \in I_0 \cap \dots \cap I_k \text{ such that } m_i < 0, \\ 0 & \text{otherwise .} \end{cases}$$

Applying Claim 8.22 to $I = I_0 \cap \dots \cap I_k$ and $V_I = V_{I_0} \cap \dots \cap V_{I_k}$, we see that $\check{C}^k(\mathcal{V}_m, \mathbb{C}) = Q_m^k$ for all m and k. □

Now we are in a position to complete our proof:

$$\begin{aligned} H^k(Q_m^\star) &\overset{\text{Claim 8.23}}{=} H^k(\mathcal{V}_m, \mathbb{C}) \\ &\overset{\text{Claim 8.20}}{=} \check{H}^k(L_m, \mathbb{C}) \\ &\overset{\text{Proposition 8.17}}{=} \check{H}^k(\text{point}). \end{aligned}$$

Hence,

$$H^{>0}(Q_m) = 0,$$

as required.

9. Singular [Q,R]=0 for non-symplectic Hamiltonian G-manifolds

Let G be a torus and let (M, ω, Φ, J) be a quantizable Hamiltonian G-manifold equipped with an equivariant stable complex structure J. Assume that M is compact and M^G is discrete. We stress that we do not require J to be compatible with ω nor ω to be non-degenerate. We have shown that [Q,R]=0 holds at all regular values α of Φ. When α is singular, there is difficulty even in *stating* this result: if ω is symplectic and J is compatible, for "singular [Q,R]=0" to hold we must desingularize by shifting, passing to $\alpha + h$ which is *inside* the moment map image $\Phi(M)$. In the degenerate case, the moment image is meaningless.

However, in the degenerate case there are two polytopes which can be used to replace $\Phi(M)$. When stated in terms of these polytopes, the [Q,R]=0 theorem continues to hold. Alternatively, [Q,R]=0 holds with a particular desingularization which is different from the Meinrenken-Sjamaar "shift desingularization". Whereas there might remain singular values $\alpha \in \mathbb{Z}_G^*$ to which the above results do not apply; they do establish versions of "singular [Q,R]=0" for most values α without assuming non-degeneracy.

Consider the non-polarized moment cones

$$C_p = \Phi(p) + \sum_{j=1}^{d} \mathbb{R}_+ \alpha_{j,p}, \qquad p \in M^G. \tag{8.58}$$

We set

$$\Delta_1 = \bigcap_p C_p \tag{8.59}$$

and

$$\Delta_2 = \operatorname{conv}\{\Phi(p) \mid p \in M^G\}. \tag{8.60}$$

The set Δ_2, being the convex hull of a finite set of points, is always a compact convex polytope. The set Δ_1 is a finite intersection of convex polyhedral cones and can be non-compact. However, if Δ_1 is compact, $\Delta_1 \subseteq \Delta_2$. (We leave the proof of this fact to the reader as an exercise.)

Recall that if ω is symplectic and J comes from an almost complex structure compatible with ω, then $\Delta_1 = \Delta_2 = \Phi(M)$. (See Section 2 of Chapter 2.) If ω is not symplectic, then often $\Delta_1 \neq \Delta_2$:

Example 8.24. Let S^1 act on $S^2 \subset \mathbb{C} \times \mathbb{R}$ by rotations with standard moment map. Consider the stable complex structure coming from

$$TS^2 \oplus \mathbb{R} \cong T(\mathbb{C} \times \mathbb{R})|_{S^2} = S^2 \times \mathbb{C} \times \mathbb{R} \subset S^2 \times \mathbb{C}^2.$$

Both isotropy weights are equal to $+1$. Hence, Δ_1 is an infinite ray and Δ_2 is an interval. On the other hand, if we equip S^2 with the stable complex structure which is the opposite from the standard complex structure, Δ_1 is empty (being the intersection of two disjoint rays pointing in opposite directions) and Δ_2 is an interval.

A close examination of the proof of Theorem 8.3 yields the following, more general, result. Let (M, ω, Φ, J) be a quantizable Hamiltonian G-manifold equipped with an equivariant stable complex structure. Assume that M is compact and M^G is discrete. Consider the moment cones (8.58), and, for each "polarizing vector" η (see Proposition 4.18), consider the "polarized moment cones"

$$C_p^\# = \Phi(p) + \sum_{j=1}^{d} \mathbb{R}_+ \alpha_{j,p}^\#, \qquad p \in M^G.$$

Let $\alpha \in \mathbb{Z}_G^*$ be a weight for G. Let $\alpha + h$ be a nearby regular value for Φ.

Proposition 8.25. *Assume that there exists a polarization such that $\alpha + th \in$ interior(C_p) if $\alpha \in C_p^\#$, for all $0 < t \leq 1$ and for each $p \in M^G$. Then*

$$\mathcal{Q}(M, \omega, \Phi, J)^\alpha = \mathcal{Q}(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h}). \tag{8.61}$$

In particular,

$$\mathcal{Q}(M, \omega, \Phi, J)^\alpha = \mathcal{Q}(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h}) = 0$$

when there exists a polarization such that $\alpha \notin C_p^\#$ for all $p \in M^G$.

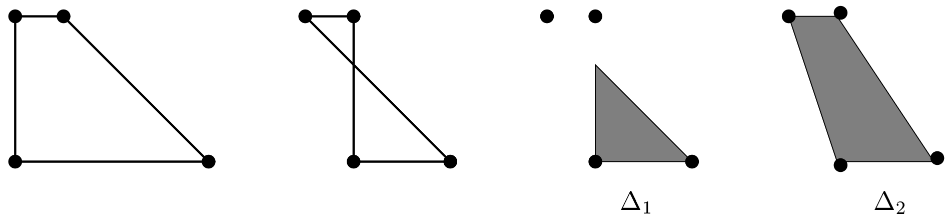

FIGURE 8.4. Generalized moment polytopes for a Hirzebruch surface

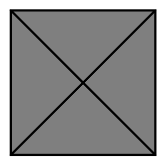

FIGURE 8.5. $\mathbb{CP}^3$ with a 2-torus action

Proposition 8.25 implies that quantization commutes with reduction for values of α that are outside Δ_2 or inside Δ_1:

PROPOSITION 8.26. *Let G be a torus and (M, ω, Φ, J) a quantizable Hamiltonian G-manifold equipped with an equivariant stable complex structure. Assume that M is compact and M^G is discrete. Let $\alpha \in Z_G^*$ be a weight for G and let $h \in \mathfrak{g}^*$ be such that $\alpha + th$ is a regular value of Φ for all $0 < t \leq 1$. If $\alpha \notin \Delta_2$,*

$$\mathcal{Q}(M, \omega, \Phi, J)^\alpha = \mathcal{Q}(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h}) = 0. \tag{8.62}$$

If $\alpha + th \in \text{interior}(\Delta_1)$ for all $0 < t \leq 1$,

$$\mathcal{Q}(M, \omega, \Phi, J)^\alpha = \mathcal{Q}(M_{\alpha+h}, \omega_\alpha, J_{\alpha+h}). \tag{8.63}$$

PROOF OF PROPOSITION 8.26. Suppose that $\alpha \notin \Delta_2$. Because Δ_2 is convex, there exists a hyperplane that strictly separates α from Δ_2. Equivalently, there exists $\eta \in \mathfrak{g}$ such that $\alpha(\eta) < \Phi^\eta(p)$ for all $p \in M^G$. Choose this η as a polarizing vector. Then $\alpha \notin C_p^\#$ for all $p \in M^G$, and, by Proposition 8.25, (8.62) holds.

Suppose that $\alpha + th \in \text{interior}(\Delta_1)$ for all $0 < t \leq 1$. Then $\alpha + th \in \text{interior}(C_p)$ for all $0 \leq t \leq 1$ and for all $p \in M^G$, regardless of whether on not $\alpha \in C_p^\#$ and, in particular, regardless of the polarization. By Proposition 8.25, (8.63) holds. □

EXAMPLE 8.27. Let M be the Kähler Hirzebruch surface corresponding to the first polytope in Figure 8.4. There exists a pre-symplectic form whose moment image is the second, "twisted" polygon in Figure 8.4. (See [**KT1**].) With this structure, Δ_1 and Δ_2 are as shown in the figure.

For α in the interior of the bottom triangle of the "twisted polygon", M_α is a point and $\dim \mathcal{Q}(M_\alpha) = 1$; for α in the interior of the top triangle, M_α is a point with orientation -1 and $\dim \mathcal{Q}(M_\alpha) = -1$.

For $\alpha \in \mathbb{Z}^2$ in the closure of the bottom triangle, $\mathcal{Q}(M)^\alpha = 1$; for $\alpha \in \mathbb{Z}^2$ in the interior of the top triangle or in the interior of its top edge, $\mathcal{Q}(M)^\alpha = -1$; for all other α's, $\mathcal{Q}(M)^\alpha = 0$. (See [**KT1**].)

In this example we see that $\mathcal{Q}(M)^\alpha$ is equal to $\mathcal{Q}(M'_\alpha)$ when the latter corresponds to the moment map Φ' for which the horizontal edges are shifted slightly outwards, the vertical edge is shifted slightly to the left, and the diagonal edge is shifted slightly to the right. This is a special case of the following result, which is proved exactly in the same way as Theorem 8.1 in Section 3.

PROPOSITION 8.28. *Let G be a torus and let $\mathbb{L}$ be a a pre-quantization line bundle over a Hamiltonian G-manifold (M, ω, Φ, J) equipped with an equivariant stable complex structure. Assume that M is compact and M^G is discrete. Fix $\alpha \in \mathbb{Z}^*_G$.*

If α is a regular value for Φ, then $[\mathrm{Q}, \mathrm{R}] = 0$ at α by Theorem 8.1.

Suppose that α is singular for Φ. Let K be the anti-canonical line bundle for J and let Φ_K be the corresponding moment map. For a small enough $r > 0$, there exists a small (possibly zero) $h' \in \mathfrak{g}^*$ such that α is a regular value for $\Phi' := \Phi + r\Phi_K - h'$. We desingularize the reduced space by passing to $(M'_\alpha, \omega'_\alpha)$, where $M'_\alpha = Z'/G$ for $Z' = (\Phi')^{-1}(\alpha)$ and $\omega' = \text{curvature}\, \mathbb{L}'_\alpha$ for

$$\mathbb{L}'_\alpha = (\mathbb{L}|'_Z \otimes \mathbb{C}_{-\alpha})\,/G.$$

With this desingularization, we have $[\mathrm{Q}, \mathrm{R}] = 0$:

$$\mathcal{Q}(M, \omega, \Phi, J)^\alpha = \dim \mathcal{Q}(M'_\alpha, \omega'_\alpha).$$

EXAMPLE 8.29. Consider $\mathbb{CP}^3$ equipped with its standard Kähler structure and the $(S^1 \times S^1)$-action with moment map shown in Figure 8.5. The middle point α is a singular value for Φ, and, moreover, for $\Phi + r\Phi_K$ for all $r \geq 0$. We can still apply Proposition 8.28, with $h' \neq 0$. Alternatively, [Q,R]=0 at α for any "shift desingularization", by Proposition 8.26, because α is in the interior of the square Δ_1.

10. Overview of the literature

In this section we give a brief (and admittedly incomplete) survey of publications related to the [Q,R]=0 conjecture. The reader interested in more detailed accounts should consult [**Sj2**] (for the results prior to 1995) and [**Ve5**].

Special cases of the [Q,R]=0 conjecture were proved in [**GS3**] and [**GS4**]. For example, in [**GS3**] this conjecture was verified for Kähler manifolds, and in [**GS4**] it was shown that a "stable" version of this conjecture is true in the sense that it holds if one replaces the symplectic manifold (M, ω) by the symplectic manifold $(M, n\omega)$ with n sufficiently large. The papers [**GS3**] and [**GS4**] appeared in the early 1980's, and then there was a ten year hiatus during which there were very few further developments (the only development we are aware of being the article [**DET**]). In 1992, however, some work of Witten on two-dimensional quantum field theory opened a new avenue of approach to this conjecture: Witten was concerned with computing the cohomology ring structure of the moduli space of vector bundles over a Riemann surface; and, to do so, invented a computational technique in equivariant de Rham theory which has become known as "non-abelian localization". (See [**Wi2**].) This was refined further by Jeffrey and Kirwan in the paper [**JK1**] (see also [**JK2, JK3**]) in which they showed how non-abelian localization is related to the usual abelian localization of Atiyah-Bott-Berline-Vergne. (For a cobordism version of their result, see [**GGK1**].) Then it was observed, simultaneously, by a number of persons (among others, Meinrenken, Jeffrey, and Kirwan) that the Jeffrey-Kirwan version of non-abelian localization enables one to give a very simple heuristic proof of [Q,R]=0. (For an account of this "proof" see [**Guil1**].) In the spring of 1995, Meinrenken [**Men1**] succeeded in making this heuristic argument rigorous, and, at about the same time, Vergne [**Ve4**] (see also [**Ve1, Ve3**]) gave an alternative proof which was also, to some extent, based on non-abelian localization. Next, in the fall of 1995, a completely different proof, using Lerman's theory of symplectic cutting, was discovered, independently, by Duistermaat-Guillemin, Meinrenken and Wu (see [**DGMW**] and, for the symplectic-cutting ideas, [**Ler1**]). In the spring of 1996, Meinrenken adapted the symplectic cutting methods of this paper to the non-abelian case and proved in [**Men2**] the non-abelian analogue of (8.1). Finally, in the fall of 1996, he and Sjamaar showed, in [**MeSj**] (see also [**Sj1**]), that the results of [**Men2**] were true for *singular* reduced spaces.

In some sense the Meinrenken–Sjamaar theorem is end of the story as far as the [Q,R]=0 conjecture is concerned. However, some further interesting developments have occurred subsequently. In [**TZ**] Tian and Zhang gave a third completely new proof of the conjecture using analytic Morse theory *à la* Witten, and we started developing in [**GGK1**] a cobordism approach to the problem, of which we gave a fuller account in this book. Also it was discovered that [Q,R]=0 could be formulated in the setting of deformation quantization, and a proof of it in this setting was given by Fedosov [**Fe**]. Finally, quite recently, Meinrenken and Woodward [**MW1**] have given a simple analytic proof of the Verlinde factorization theorem which can be interpreted as a version of [Q,R]=0 for loop groups acting on infinite dimensional symplectic manifolds.

Essentially topological proofs of the abelian case of [Q,R]=0 were obtained in [**CKT, Met3, Par2**]. For instance, in [**Par2**] the [Q,R]=0 theorem is proved for regular values of abstract moment maps for torus actions (and for regular and singular values for compact connected Lie groups acting on symplectic manifolds). We also refer the reader to [**Brav4**] for a non-compact version of the [Q,R]=0 theorem and to [**Te**] for a proof of [Q,R]=0 for smooth compact polarized varieties.