APPENDIX H

Characteristic numbers, non-degenerate cobordisms, and non-virtual quantization

1. The Hamiltonian cobordism ring and characteristic classes

In this section we discuss the problem of calculating the Hamiltonian cobordism ring. We start this discussion with compact cobordisms of compact G-manifolds. Throughout this section all manifolds are assumed to be stable complex[1] and, as usual, G is a torus.

1.1. Formulation of the problem. Recall that a compact stable complex Hamiltonian G-manifold is a stable complex G-manifold equipped with a cohomology class $u \in H^2_G(W)$. The Hamiltonian cobordism ring $\mathcal{H}^G_*$ of such manifolds is very large, and probably does not have an elegant description in terms of generators and relations. Here, however, we will be more interested in the invariants of cobordism. Hence, our focus will be on the "dual" problem of finding a full collection of invariants of Hamiltonian cobordism. A satisfactory solution for this problem is unknown also, but partial information is available.

Since in this book we are mainly interested in mixed characteristic numbers (see Section 2 for the definition), we will concentrate on the following questions:

(A) Do mixed characteristic numbers form a full system of invariants of Hamiltonian cobordism?
(B) What are the relations among the mixed characteristic numbers of cobordism classes in a given dimension?
(C) Which cobordism classes of Hamiltonian G-manifolds are represented by symplectic manifolds? Are there additional constraints on the mixed characteristic numbers of symplectic manifolds?

Complete answers to these questions are known only in the non-equivariant case, when the group is trivial and the moment map is zero. Let us discuss this case.

1.2. Non-equivariant Hamiltonian cobordisms.

The ring of non-equivariant Hamiltonian cobordisms with zero cohomology classes is the same as the ring of (stable) complex bordisms MU_*, [**MiSt**]. As a ring, $\mathrm{MU}_* \otimes \mathbb{Q}$ is freely generated by complex projective spaces. The set of generators of MU_* itself is also known, [**MiSt**]. The mixed characteristic numbers become in this case the ordinary Chern numbers. They form a a complete set of independent invariants of MU_* and they may assume arbitrary rational values on $\mathrm{MU}_* \otimes \mathbb{Q}$, [**MiSt**].

[1] A complex cobordism theory is always simpler than a real one. This is true already in the non-equivariant case; see [**MiSt**].

Turning to general non-equivariant Hamiltonian cobordisms, for simplicity, we restrict our attention to the case where the cohomology classes are integral. Then the theory of Hamiltonian cobordisms becomes identical with the theory of stable complex cobordisms equipped with integral cohomology classes of degree two. The resulting cobordism ring can be identified with the stable complex bordism ring $\mathrm{MU}_*(\mathbb{CP}^\infty)$ of $\mathbb{CP}^\infty$ for which MU_* is the "coefficient ring". The ring $\mathrm{MU}_*(\mathbb{CP}^\infty)$ is calculated in [**Sto**]. Its free (additive) generators are the homotopy classes of the maps of the generators of MU_* to $\mathbb{CP}^\infty$. As above, the mixed characteristic classes form a full system of independent invariants of integral non-equivariant Hamiltonian cobordism, [**Sto**], and they can assume arbitrary values over $\mathbb{Q}$. This answers questions (A) and (B) in this case.

One can also show that every integral non-equivariant Hamiltonian manifold is cobordant to a symplectic manifold, [**Bak, Gin2, Mor**].[2] (For example, a stable complex manifold equipped with the zero cohomology class is still cobordant to a symplectic manifold, which necessarily has more than one connected component.) In other words, symplectic manifolds generate $\mathrm{MU}_*(\mathbb{CP}^\infty)$. Hence, the mixed characteristic numbers form a full system of independent invariants of Hamiltonian cobordism classes of symplectic manifolds. By "dualizing" we also conclude that every collection of mixed characteristic numbers which is possible for an integral non-equivariant Hamiltonian manifold is also attainable for an integral symplectic manifold.

1.3. The equivariant cobordism ring and the Hamiltonian cobordism ring. The questions (A)–(C) become considerably more subtle in the equivariant case. As above, the key to understanding question (A) is the stable complex equivariant cobordism ring $\mathcal{U}_*^G$, i.e., the ring of Hamiltonian cobordisms in which equivariant cohomology classes are assumed to be zero. This ring plays the role of the coefficient ring for the theory of stable complex Hamiltonian cobordisms.

To avoid possible confusion we note that there are two essentially different G-equivariant (compact) stable complex cobordism theories. One theory is the naive geometric theory of G-cobordisms defined as above with manifolds equipped with (tangential) G-equivariant stable complex structures. This theory arises in applications we are interested in and is the only one considered in this book and $\mathcal{U}_*^G$ denotes the ring of such cobordisms. However, this theory is relatively poorly understood.[3] The second theory, introduced by tom Dieck, [**tD**], is a certain "extension" of the geometric theory by homotopy theoretical means. This theory is accessible by topological methods and is much better understood; see, e.g., [**tD, Kr, May, Si1, Si2**] and references therein. This theory is however less suitable for our goals.

Although a complete proof is unavailable at the moment, there is some evidence in favor of the conjecture that the answer to question (A) is affirmative for equivariant cobordisms: equivariant Chern numbers distinguish equivariant cobordism classes. For G-manifolds with isolated fixed points this is established in Section 3. A relevant result in the homotopy theoretical setting is obtained by Sinha in [**Si1**]. Note that a positive answer to (A) for $\mathbb{T}^n$-equivariant cobordisms for all n would imply the affirmative answer to that question for stable complex Hamiltonian cobordisms; see Section 2.

[2] We emphasize again that the orientations of symplectic manifolds are not assumed to be induced by the symplectic forms.

[3] For some discrete groups (e.g., $\mathbb{Z}_p$) the real analogues of the ring $\mathcal{R}$ are described in [**CF3**].

In contrast with the non-equivariant setting, the answer to question (B) is unknown already for equivariant cobordisms. In this connection we note that in every even dimension there are non-vanishing equivariant Chern classes of arbitrarily high degrees. By the localization theorem, the corresponding Chern numbers can be expressed through the (cobordism classes of) fixed point data. However, the group of these cobordism classes is infinitely generated in every dimension. Hence, there is no reason to expect that relations exist. For example, in dimension two the only non-trivial Chern classes are powers of the first Chern class and there appear to be no relations among the Chern numbers.

The answer to question (C) is also unknown. In the non-equivariant case the argument is based on a complete understanding of the generators of the cobordism ring, lacking for the equivariant cobordism ring. Yet it is not unreasonable to conjecture that modulo torsion the Hamiltonian cobordism ring is generated by symplectic manifolds when G is a torus.

From the symplectic geometry perspective, the main reason to consider Hamiltonian cobordisms is that mixed characteristic numbers are invariants of Hamiltonian cobordism. To summarize the above discussion, we can say that the theory of Hamiltonian cobordisms is likely to be the "minimal" theory with this property. Furthermore, once we are only concerned with such integral invariants, there is no reason to restrict one's attention to symplectic manifolds. Moreover, as we have already seen, non-symplectic Hamiltonian manifolds arise naturally already in symplectic geometry problems.

We conclude this discussion by mentioning that the theory of Hamiltonian cobordisms and its applications to symplectic geometry have been inspired by the theory of non-degenerate cobordisms to which we will turn in Section 4.

1.4. Relations between compact and non-compact cobordisms. In this book we consider *proper* Hamiltonian cobordisms, where non-compact manifolds are allowed but all moment maps are assumed to be proper. In particular, we then work with the equivariant forms $\omega - \Phi$ rather than their cohomology classes $[\omega - \Phi]$. Furthermore, we may discard the two-forms and work with *abstract moment maps* (see Chapter 3).

As we have already mentioned, the main problem which arises when one works with non-compact cobordisms is that compact manifolds may all turn out to be cobordant to zero in the resulting cobordism theory. This is certainly very undesirable and it is important to know that this does not happen when we pass to proper cobordisms of abstract moment maps.

Thus let us compare G-equivariant compact cobordisms and proper cobordisms of abstract moment maps. Note that any compact G-manifold or cobordism can be equipped with the zero abstract moment map which is automatically proper due to compactness. In this way we obtain a map from the G-equivariant compact cobordism theory to the theory of proper cobordisms of abstract moment maps. The main result of [**GGK2**] is that this map is one-to-one, i.e., the compact theory is embedded into the proper one. In other words, introducing non-compact cobordisms does not result in new relations between compact G-manifolds.

The linearization theorem (Theorem 4.12), comes close to a satisfactory description of the non-compact theory. Using reduction, we also conclude that fiber bundles of toric orbifolds (with possibly exotic stable complex structures) generate the image of the compact G-equivariant cobordism theory in the compact equivariant orbifold cobordism theory. (Also see [**Mart1**].)

The question of which non-compact manifolds are cobordant to compact ones is still open. In other words, as of today, the image of the inclusion of the compact theory into the proper one is unknown. The problem is closely related, via the localization formula, to describing relations among the representations of G on the normal bundles to M^G when M is compact. For $G = S^1$, the relations are found by Gusein-Zade, [**GZ2**]. However, this description is rather complicated and difficult to decipher, even in the case of isolated fixed points. When G is a torus, the problem of describing the set of relations is completely open.

An important ingredient of the theory of compact cobordisms which does not have an immediate extension to the non-compact case is the notion of characteristic numbers. Although one can sometimes define characteristic numbers in the non-compact case using, for example, the cutting construction, reduction, or the localization theorem, it is not clear whether the resulting invariants are useful and tangible.

2. Characteristic numbers

In this section we focus on characteristic numbers for the ring of G-equivariant (stable) complex cobordisms, $\mathcal{U}_*^G$, and the ring of (stable) complex Hamiltonian cobordisms of G-manifolds, $\mathcal{H}_*^G$. All Hamiltonian G-manifolds in this sections are assumed to be integral, i.e., the corresponding equivariant cohomology classes $\frac{1}{2\pi}[\omega - \Phi]$ are assumed to be integral.

The *equivariant characteristic numbers* $c_I^G(W)$ of a compact oriented stable complex G-manifold are defined as

$$c_I^G(W) = \left\langle (c_1^G)^{i_1} \cdots (c_k^G)^{i_k}, [W] \right\rangle \in H^*(BG),$$

where $I = (i_1, \ldots, i_k)$ is a multi-index and c_l^G is the lth equivariant Chern class of W. For a stable complex Hamiltonian G-manifold W, the *mixed equivariant characteristic numbers* $c_{j,I}^G(W)$ are defined by

$$c_{j,I}^G(W) = \left\langle u^j \cdot (c_1^G)^{i_1} \cdots (c_k^G)^{i_k}, [W] \right\rangle \in H^*(BG), \tag{H.1}$$

where u is the equivariant cohomology class of degree two. Note these characteristic numbers can be non-zero if the degree of the product $(c_1^G)^{i_1} \cdots (c_k^G)^{i_k}$ or $u^j \cdot (c_1^G)^{i_1} \cdots (c_k^G)^{i_k}$ is greater than $\dim W/2$.

In the next section we will present some evidence supporting the conjecture that the equivariant characteristic numbers separate elements of $\mathcal{U}_*^G$ when G is a torus. Here we will construct a monomorphism $\Sigma_G \colon \mathcal{H}_*^G \to \mathcal{U}_{*+2}^G$ and show that if this conjecture holds, then mixed characteristic numbers also separate stable complex Hamiltonian equivariant cobordisms.

By Theorem C.47, for a given integral class $u \in H_G^2(W; \mathbb{Z})$, there exists a G-equivariant principle U(1)-bundle $E \to W$. (Note that, in contrast with the non-equivariant case, this is a non-trivial fact.) Moreover, this bundle is unique up to an equivariant principle bundle isomorphism. Let $\overline{W} \to W$ be the S^2-bundle associated with $E \to W$, where S^2 is equipped with the standard U(1)-action by rotations about the z-axis. The G-action on $\overline{W}$ inherited from E commutes with the natural right $\mathrm{U}(1) = S^1$-action. Hence, $\overline{W}$ can be viewed as a stable complex

$(G \times S^1)$-manifold. It is not hard to see that the mapping $(W, u) \to \overline{W}$ gives rise to a linear homomorphism $\Sigma_G \colon \mathcal{H}^G_* \to \mathcal{U}^{G\times S^1}_{*+2}$.

Proposition H.1. *The homomorphism Σ_G is a monomorphism. Furthermore, the $G \times S^1$-characteristic numbers of $\overline{W}$ can be expressed via the mixed G-characteristic numbers of (W, u).*

Proof. Note that W is canonically embedded in $\overline{W}$ as the "North Pole section", i.e., as the connected component of $\overline{W}^{S^1}$ with the representation of S^1 on the normal bundle isomorphic to the standard representation of S^1 on $\mathbb{C}$. (The "South Pole section" has the complex conjugate normal representation.) The class u is then the first G-equivariant Chern class of the normal bundle to W in $\overline{W}$.

Let (W_0, u_0) and (W_1, u_1) be oriented stable complex G-manifolds equipped with equivariant cohomology classes. Assume that $\overline{W_0}$ is $G \times S^1$-cobordant to $\overline{W_1}$. Then $\overline{W_0}^{S^1}$ is G-cobordant to $\overline{W_1}^{S^1}$. Moreover, the same is true for the components with a given normal representation of S^1. Thus W_0 is cobordant to W_1 as an oriented stable complex G-manifold. Finally, the cobordism between the cohomology classes is given by the first G-equivariant Chern class of the normal bundle. This proves that Σ_G is injective.

The second assertion follows from the appropriate version of the Atiyah–Bott–Berline–Vergne localization theorem (cf. Section 7 of Appendix C). ☐

Remark H.2. As is clear from the proof, the theorem holds for any compact Lie group G.

Example H.3. The group $\mathcal{U}^{S^1}_2$ is the direct sum of the following summands.

- The group of (stable) complex cobordisms of two-manifolds with trivial S^1-action. This group is isomorphic to $\mathbb{Z}$, generated by S^2, and the only invariant is the first Chern number (see [**MiSt**]).
- The free abelian group generated by S^2 with the standard S^1-action.
- For every non-zero integer $k \neq 1$, the free abelian group generated by S^2 with the S^1-action which is obtained from the standard one via the k-fold covering $S^1 \to S^1$. (The trivial action would correspond to $k = 0$ and the standard action would result from $k = 1$.)

A similar description applies when $G = \mathrm{SU}(2)$ except that in this case one does not have components corresponding to the k-fold coverings.

3. Characteristic numbers as a full system of invariants

Our goal in this section is to show that two stable complex compact G-manifolds with isolated fixed points are cobordant if and only if their equivariant characteristic numbers are equal. Here, as usual, G is a torus.

In one direction the assertion is clear: by Stokes' formula, cobordant manifolds have equal characteristic numbers. Hence we need to prove

Theorem H.4. *Let G be a torus and let W be a compact stable complex G-manifold such that W^G is discrete and the mixed characteristic numbers of W are equal to zero. Then W is cobordant to zero.*

Remark H.5. As we have mentioned above, the condition that the fixed points are isolated is probably purely technical. Conjecturally, Theorem H.4 holds without this assumption.

PROOF OF THEOREM H.4. The tangent spaces T_pW, for $p \in W^G$, are stable complex representations of G. Moreover, these vector spaces are in fact genuine complex representations of G, as readily follows from the definition of equivariant stable complex structures (see Section 1 of Appendix D). In addition, the spaces T_pW inherit orientations from W, and these orientations may differ from the orientations induced by the complex structures. Let m_V be the number of times with which a complex representation V of G occurs among the representations T_pW, for $p \in W^G$. Note that m_V is understood algebraically:

$$m_V = \#\{p \in W^G \mid T_pW = V\} - \#\{p \in W^G \mid T_pW = -V\},$$

where $T_pW = -V$ means that the two representations are equal, but the orientations are different.

LEMMA H.6. $m_V = 0$ *for every complex representation* V *of* G.

Let us postpone the proof of the lemma and first finish the proof of the theorem. This can be done directly using the methods of [**GZ1**]. The following argument relies on the linearization theorem and the results of [**GGK2**].

Let us equip W with an abstract moment map which is identically zero and "polarize" by picking an arbitrary vector $\eta \in \mathfrak{g}^*$ such that the zero set $M^\eta = \{\eta_M = 0\}$ coincides with the fixed point set M^G.

By the linearization theorem for abstract moment maps (Theorem 4.12), we have

$$W \sim \bigsqcup_{p \in W^G} T_pW \tag{H.2}$$

where the tangent spaces T_pW are equipped with the induced G-actions, orientations, and complex structures. The tangent spaces T_pW are also equipped with abstract moment maps Φ_p which are polarized and equal to zero at the origin. As is shown in [**GGK2**], the theory of compact cobordisms is embedded into the theory of proper cobordisms of abstract moment maps. Thus it suffices to show that the right-hand side of (H.2) is cobordant to zero.

Note that two quadratic abstract moment maps on the same representation of G, which are polarized with respect to the same vector, are cobordant if and only if these moment maps assume equal values at the origin. (This can be easily proved in the same manner as Lemma 3.31.) Since $\Phi_p(0) = 0$ for all $p \in W^G$, the contribution of T_pW to the cobordism class of the right-hand side of (H.2) is entirely determined by the complex representation of G on T_pW and the orientation of T_pW and is independent of the moment map Φ_p. Therefore, the right-hand side of (H.2) is cobordant to zero if and only if every representation V occurs in it with zero multiplicity, i.e., $m_V = 0$. Hence the theorem follows from Lemma H.6.

PROOF OF LEMMA H.6. A representation of G on $\mathbb{C}^n$ is determined by its weights $\alpha_1, \dots, \alpha_n$, which are elements of $\mathfrak{g}^*$. The weights, in turn, are determined by the values $\sigma_1(\alpha_1, \dots, \alpha_n), \dots, \sigma_n(\alpha_1, \dots, \alpha_n)$, where $\sigma_1, \dots, \sigma_n$ are the elementary symmetric functions. Since $\alpha_1, \dots, \alpha_n$ are linear functions on $\mathfrak{g}$, the symmetric function $\sigma_k(\alpha_1, \dots, \alpha_n)$ is a polynomial on $\mathfrak{g}$ of degree k. Denote by $\sigma(p) = (\sigma_1(p), \dots, \sigma_n(p))$ the collection of such symmetric functions for the representation of G on T_pW, $p \in W^G$. Clearly, $\sigma(p)$ determines the representation, but not the orientation of T_pW inherited from W. For a collection σ of n polynomials on $\mathfrak{g}$, let m_σ be the algebraic number of representations T_pW with $\sigma(p) = \sigma$. (Of course, we set $m_\sigma = 0$ if σ does not occur as the collection of symmetric functions for any of the T_pW.) In other words,

$$m_\sigma = m_V,$$

where the collection of symmetric functions for V is σ. Thus our goal is to show that $m_\sigma = 0$ for any σ.

The restriction of the kth equivariant Chern class $c_k^G(TW)$ of W to p is equal to the kth equivariant Chern class $c_k^G(T_pW)$ of the vector bundle $T_pW \to p$ which in turn is $\sigma_k(p)$:

$$c_k^G(TW)\mid_p = c_k^G(T_pW) = \sigma_k(p) \in S^k(\mathfrak{g}).$$

The normal bundle to p in W is T_pW with the orientation inherited from W. The equivariant Euler class of this bundle is $\pm c_n(T_pW) = \pm\sigma_n(p)$, where the sign is positive if the orientation of T_pW agrees with the complex orientation and negative otherwise.

Thus using the Atiyah–Bott–Berline–Vergne localization theorem (Theorem C.53), we obtain the following expression for the equivariant Chern numbers of M:

$$\begin{aligned} c_I^G(W) &= \left\langle c_1^G(TW)^{i_1} \cdot \ldots \cdot c_n^G(TW)^{i_n}, [W] \right\rangle \\ &= \sum_{p \in W^G} \pm \frac{\sigma_1(p)^{i_1} \cdot \ldots \cdot \sigma_n(p)^{i_n}}{\sigma_n(p)} \\ &= \sum_{p \in W^G} \pm\sigma_1(p)^{i_1} \cdot \ldots \cdot \sigma_n(p)^{i_n - 1} \\ &= \sum_\sigma m_\sigma \sigma_1^{i_1} \cdot \ldots \cdot \sigma_n^{i_n-1}, \end{aligned}$$

where $I = (i_1, \ldots, i_n)$ is a multi-index. In what follows we will always assume that $i_n \geq 1$ to make every term on the right-hand side polynomial.

By the hypothesis of the theorem, $c_I^G(W) = 0$ for all I. To show that $m_\sigma = 0$ for any σ, we will make suitable choices of multi-indices I.

Let $\eta_1, \ldots, \eta_s$ be all the distinct polynomials that occur as $\sigma_1(p)$ for $p \in W^G$. Then

$$\begin{aligned} c_I^G(W) &= \sum_{j=1}^s \sum_{\substack{\sigma \\ \sigma_1 = \eta_j}} m_\sigma \sigma_1^{i_1} \cdot \ldots \cdot \sigma_n^{i_n-1} \\ &= \sum_{j=1}^s \eta_j^{i_1} \sum_{\substack{\sigma \\ \sigma_1 = \eta_j}} m_\sigma \sigma_2^{i_2} \cdot \ldots \cdot \sigma_n^{i_n-1} \\ &= 0. \end{aligned}$$

Let i_1 vary in the range $0, \ldots, s-1$. Then $(\eta_j^{i_1})$ is a van der Monde matrix (with polynomial entries). Hence,

$$\sum_{\substack{\sigma \\ \sigma_1 = \eta_j}} m_\sigma \sigma_2^{i_2} \cdot \ldots \cdot \sigma_n^{i_n-1} = 0$$

for all j and any multi-index $(i_2, \ldots, i_n)$. Note that if σ is such that $\sigma_1 \neq \eta_j$ for any j, we automatically have $m_\sigma = 0$. So the last identity can also be written as

$$\sum_\sigma m_\sigma \sigma_2^{i_2} \cdot \ldots \cdot \sigma_n^{i_n-1} = 0$$

for any multi-index $(i_2, \dots, i_n)$, where we sum over $\sigma = (\sigma_2, \dots, \sigma_n)$. Proceeding inductively, we conclude that $m_\sigma = 0$ for any σ. □

□

REMARK H.7. In the proof of the theorem, to conclude that M is cobordant to zero, we used only a finite number of different characteristic classes. The exact number needed depends on the dimension and the size of the fixed point set.

4. Non-degenerate cobordisms

One of the themes of this book stems from the fact that geometric quantization is an invariant of cobordism. For the major part of the book, both cobordism and geometric quantization are understood as formal objects. Cobordism is stable–complex equivariant cobordism of manifolds equipped in addition with an equivariant cohomology class of degree two. Geometric quantization is understood as a virtual representation given for example by the Riemann–Roch formula.

In this section we show that, at least in the non-equivariant case, geometric quantization remains, in some sense, an invariant of cobordism when both quantization and cobordism are defined more geometrically.

Let us first recall the definition of *non-degenerate cobordism* of symplectic manifolds given in [**Gin1, Gin2**].[4] Let (W_0, ω_0) and (W_1, ω_1) be compact oriented (integral) symplectic manifolds of dimension $2n$. Note that the orientations of W_0 and W_1 are not assumed to be induced by the symplectic structures. We say that (W_0, ω_0) and (W_1, ω_1) are cobordant if there exists an oriented cobordism M between W_0 and W_1 equipped with a closed (integral) two-form σ of maximal rank (equal to $2n$), which restricts to ω_0 and ω_1 on W_0 and, respectively, W_1.

We emphasize again that this notion of cobordism is different from the ones used in other parts of this book.

REMARK H.8. Since σ has maximal rank on an odd–dimensional manifold, the null space $\ker\sigma$ is one–dimensional at every point of M. The integral curves of the line field $\ker\sigma$ are called the *characteristics* of σ. The orientation of M together with σ^n give rise to the orientation of $\ker\sigma$ and hence to the orientations of the characteristics. For almost all $x \in M$, the characteristic through x begins and ends on ∂M. There may be, however, characteristics that do not have beginnings and/or ends on the boundary. Similarly, for a given connected component W of ∂M, either every $x \in W$ is the beginning of a characteristic or its end, depending on whether $\int_W \omega^n$ is positive or negative. For almost all $x \in W$, the characteristic through x reaches some other component of ∂M.

As usual, cobordism equivalence classes form a ring. In what follows we will restrict our attention to the case where the symplectic forms and cobording two-forms are assumed to have integral cohomology classes. In this case, the cobordism ring, denoted by $\mathcal{B}_*$, has been calculated.

EXAMPLE H.9. Let (Σ_g, k) be the surface of genus $g \geq 0$ with symplectic area equal to k. It is not hard to show (see [**Gin1**]) that

$$(\Sigma_{g_1}, k_1) + (\Sigma_{g_2}, k_2) \sim (\Sigma_{g_1+g_2-1}, k_1 + k_2),$$

where $\sim$ denotes non-degenerate cobordism and $k_1 > 0$ and $k_2 > 0$.

[4] Throughout this section, all manifolds are assumed to be compact.

Furthermore, $\mathcal{B}_2$ is a free abelian group of rank 2. The generators are the sphere and the torus of unit area. For example, a surface of genus two is cobordant to the difference of a torus and a sphere. A full system of independent invariants in dimension two is given by the total area $\int_W \omega$ of a symplectic surface (W, ω) and the twisted Euler characteristic

$$\chi(W, \omega) = \sum \operatorname{sign}\Big(\int_{W_i} \omega\Big)\chi(W_i),$$

where W_i are the connected components of W. An elementary proof of this fact can be found in [**Gin1**]. Note that these invariants cannot assume arbitrary integer values: $\chi(W, \omega)$ is necessarily even.

To describe $\mathcal{B}_*$ note that there is a forgetful homomorphism F from $\mathcal{B}_*$ to the ring of stable complex cobordisms equipped with integral cohomology classes of second degree. This homomorphism sends the cobordism class of (W, ω) to the cobordism class of the triple $(W, J, [\omega])$, where J is an almost complex structure on W compatible with ω. The ring of such stable complex cobordisms is naturally isomorphic to the ring of bordisms $\mathrm{MU}_*(B\,\mathrm{U}(1))$, where $B\,\mathrm{U}(1) = \mathbb{CP}^\infty$.

Theorem H.10 ([**Bak, Gin2, Mor**]). *The forgetful homomorphism* $F\colon \mathcal{B}_* \to \mathrm{MU}_*(B\,\mathrm{U}(1))$ *is an isomorphism.*

Recall that for any multi-index $I = (i_1, \dots, i_k)$, the Ith mixed characteristic number of (W^{2n}, ω) is by definition

$$c_I(W) = \Big\langle [\omega]^{n-\|I\|} c_1^{i_1} \dots c_k^{i_k}, [W] \Big\rangle, \tag{H.3}$$

where c_i is the ith Chern class of W with respect to J, as usual, $[W]$ is the fundamental class of W with respect to the given orientation, and $\| I \| = 1i_1 + 2i_2 + \dots + ki_k$ is the weight of I. Note that $c_I(W) = 0$ automatically whenever $\| I \| > n$. It follows from Theorem H.10 (see [**Sto**]) that the mixed characteristic numbers form a full system of independent invariants of cobordisms of symplectic manifolds. In other words, the evaluation homomorphism $\{c_I\}\colon \mathcal{B}_{2n} \to \mathbb{Z}^{k(n)}$, where $k(n)$ is the number of multi-indices I with $\| I \| \leq n$, is one-to-one and its image has a full rank. Note, however, that this homomorphism is not onto (see Example H.9).

The assertion that F is a monomorphism is an easy consequence (see [**Gin2**]) of an appropriate h-principle, [**Grom, McD1**]. Then, using the fact that the ring of stable complex cobordisms is generated over the rationals by projective spaces, it is not hard to show that F and $\{c_I\}$ are isomorphisms modulo torsion, [**Gin2**]. The proof of the fact that F is a genuine epimorphism is more involved; see [**Bak, Mor**]. It is worth emphasizing that the non-degenerate cobordism of symplectic manifolds considered in this section is "soft"—it does not capture any deep symplectic invariants of (W, ω) beyond those determined by J and $[\omega]$. This is exactly the reason why the calculation of the ring $\mathcal{B}_*$ can be relatively easily reduced to an algebraic topological problem.

In conclusion note that a cobordism (M, σ) between (W_0, ω_0) and (W_1, ω_1) gives rise to a Lagrangian relation L in $(W_0 \times W_1, \omega_0 \oplus (-\omega_1))$: by definition, $(x, y) \in L$ if and only if x and y are connected by a characteristic of σ. According to a general principle (see., e.g., [**Wei2**]) Lagrangian relations should, in turn, induce operators between quantization spaces. Later on we will see that the cobordism (M, σ) does induce a homomorphism from the geometric quantization of (W_0, ω_0)

to the geometric quantization of (W_1, ω_1). However, defining this homomorphism, as well defining the geometric quantization, requires fixing an additional structure.

5. Geometric quantization

5.1. Two constructions of geometric quantization. The virtual geometric quantization of a symplectic manifold (W, ω) is, by definition, the index $\mathcal{Q}(W, \omega) = \ker D - \operatorname{coker} D$ of a suitable elliptic operator D (see Chapter 6). When there is no Hamiltonian group action involved, the index is just a virtual vector space, i.e., it is determined by the integer equal to its virtual dimension and given by the Riemann–Roch formula. Thus, in the non-equivariant situation, the virtual geometric quantization is a rather weak invariant of a symplectic manifold (similar to the Euler characteristic or the signature in algebraic topology).

However, one can associate to a given symplectic manifold geometric quantizations which are genuine, not only virtual, vector spaces. Here we consider two constructions of such spaces, both based on various versions of vanishing theorems according to which $\operatorname{coker} D = 0$ if the symplectic form is "large enough". Thus, by definition, the geometric quantization is just $\ker D$. The dimension of the space is, of course, "correct", i.e., equal to the index of D. An essential feature of both of these constructions is that the resulting space depends on additional data and is not canonically determined by the symplectic structure alone.

Let us give some more details on these constructions. Let (W, ω) be a symplectic manifold. Fix an almost complex structure J on W, compatible with ω. As shown in Section 6 of Chapter 6, the pair (J, ω) gives rise to a first order elliptic operator D, the rolled-up Dolbeault operator. Consider now the symplectic structures $k\omega$, $k = 1, 2, \ldots$ on W, with J fixed, and let D_k be the family of corresponding elliptic operators. As shown in [**GU**], $\operatorname{coker} D_k = 0$ if k is large enough. Thus, for such k, the quantization $\mathcal{Q}(W, k\omega) = \ker D_k$ is a genuine vector space. It should be noted, however, that the required value of k may depend on J as well as on ω.

The second construction is similar to the one which we just described but uses the Spin^c-Dirac operator in place of D. Now the initial data is a Spin^c-structure on W. For example, one can start again with an almost complex structure J compatible with ω, associate to the pair $(k\omega, J)$ a Spin^c-structure (see Appendix D) and consider the corresponding Spin^c-Dirac operator D_k. Again, $\operatorname{coker} D_k = 0$ when k is large enough, [**BU**], and we just have $\mathcal{Q}(W, k\omega) = \ker D_k$. (See also [**Brav1**].)

It is worth noting that the elliptic operators used in these two constructions *differ* from each other (although they do have equal (leading) symbols). As a result, in spite of their similarity, the constructions give, strictly speaking, *different* vector spaces, which have, of course, equal dimensions. Note also that although the orientations of the manifolds do not explicitly enter either of the constructions, the manifolds should be oriented via the symplectic forms, for then the results are consistent with the Riemann–Roch formula. We refer the reader to Chapter 6 for a more detailed discussion of these quantization constructions.

5.2. Dependence on the almost complex structure. Both methods of obtaining a genuine vector space as a quantization rely on additional structures, such as an almost complex structure J compatible with ω. The quantization space $\mathcal{Q}(W, k\omega)$ depends on J. To emphasize this dependence, let us fix W and ω and denote the quantization space, resulting from either of the two constructions, by $\mathcal{Q}_k(J)$.

Let us now examine closer the dependence of $\mathcal{Q}_k(J)$ on J, following [**GM**]. (For a completely different approach to the problem in the case of linear polarizations of $\mathbb{C}^d$ see [**Ki4**, Section 4.2] and references therein.) Let $\mathcal{J}$ be the space of almost complex structures compatible with ω. The collection of spaces $\mathcal{Q}_k(J)$ can be viewed as a vector bundle over $\mathcal{J}$. Literally, this is not accurate: the value of k required for the vanishing theorem to apply depends on J. However, the space $\mathcal{J}$ is exhausted by open subsets U_k such that the spaces $\mathcal{Q}_k(J)$ form a vector bundle $\mathcal{V}_k$ over U_k, for each k.

REMARK H.11. Sometimes one can choose k so that $U_k = \mathcal{J}$, that is, the bundle $\mathcal{V}_k$ is defined over the entire space $\mathcal{J}$. This is the case, for example, when $\dim W = 2$.

If we could identify the quantization spaces $\mathcal{Q}_k(J)$ with each other in some natural way, which only depends on the symplectic structure, they would fit together into one single "canonical" quantization, independent of a choice of J. Therefore, we would like to choose a flat connection on the bundle $\mathcal{V}_k$ which is preserved under the group of symplectomorphisms. Let us explain this more precisely.

The group of symplectomorphisms acts on the space $\mathcal{J}$ and on each of the open sets U_k. However, this action does not lift in a canonical way to the bundles $\mathcal{V}_k$. To lift a symplectomorphism to $\mathcal{V}_k$, it suffices to lift it to a contactomorphism of the pre-quantization circle bundle $E \to W$, which requires fixing a Hamiltonian. To be more specific, let $\mathcal{H}$ be the group of Hamiltonian symplectomorphisms, i.e., of symplectomorphisms generated by time-dependent Hamiltonians. Fix a pre-quantization circle bundle $E \to W$ with a connection form Θ. (See Section 2 of Chapter 6 for the definitions.) Recall that Θ is a contact form and thus the connection $\ker \Theta$ is a contact structure on E. Let $\mathcal{C}$ be the identity connected component in the group of all contactomorphisms of $\ker \Theta$. It is easy to see that there is an epimorphisms $\mathcal{C} \to \mathcal{H}$ and that, in fact, $\mathcal{C}$ is an extension of $\mathcal{H}$ by S^1.

By definition, the group $\mathcal{C}$ acts on W through the action of $\mathcal{H}$. This action canonically lifts to the original action of $\mathcal{C}$ on E. As a result, we obtain the induced action of $\mathcal{C}$ on $\mathcal{J}$ (and on $\mathcal{U}_k$) and its lift to $\mathcal{V}_k$.

One way to say that quantization is independent of J is to find a *natural* flat (or projectively flat) connection on $\mathcal{V}_k$. By a natural connection we mean one that is $\mathcal{C}$-invariant. The following result shows that for many symplectic manifolds W the bundle $\mathcal{V}_k$ does not admit a natural flat connection.

THEOREM H.12 ([**GM**]). *Let J_0 be an almost complex structure such that the stabilizer G of J_0 in $\mathcal{H}$ has positive dimension and the infinitesimal representation of G on $\mathcal{Q}_k(J_0)$ is non-trivial. Then there is no natural projectively flat connection on $\mathcal{V}_k$.*

REMARK H.13. The theorem is a more or less an immediate consequence, already on the infinitesimal level, of the following well–known fact. Denote by A the Lie algebra $C_c^\infty(W)$ of compactly supported functions on a (not necessarily compact) symplectic manifold W. Then *the commutant $\{A, A\}$ is the only ideal of finite codimension in A.* This result, closely related to the Gröenewald–van Hove theorem, holds for many other Poisson algebras; see, e.g., [**GM, GGG**] for further references. Theorem H.12 follows from the observation that if the connection existed, it would result in a non-trivial finite–dimensional projective representation of A. The kernel of this representation would be an ideal of finite codimension, which is different from the commutant.

REMARK H.14. Theorem H.12 holds already for the restriction of $\mathcal{V}$ to a neighborhood of J_0 in the $\mathcal{H}$-orbit of J_0. Note that if J_0 is Kähler, the entire orbit is comprised of Kähler complex structures. It would be interesting to see if the theorem remains correct for any compact symplectic manifold, i.e., without the additional assumption on the stabilizer G.

EXAMPLE H.15. The simplest manifold to which Theorem H.12 applies is $W = S^2$ with ω being the standard symplectic form and J_0 being the standard complex structure. In this case one can take as $\mathcal{U}$ the entire $\mathcal{H}$-orbit of J_0. The stabilizer G is SU(2) and its representation on $\mathcal{Q}_k(J_0)$ is just the standard representation of SU(2) on the space of homogeneous polynomials of degree k on $\mathbb{C}^2$. More generally, the theorem applies to coadjoint orbits of compact Lie groups.

REMARK H.16. In conclusion note that usually the bundle $\mathcal{V} = \mathcal{V}_k$ over $\mathcal{U} = \mathcal{U}_k$ admits many natural connections, which are, of course, non-flat. The following connection ∇ seems to be of particular interest. To define ∇ observe first that for any J there exists a natural inclusion $\mathcal{Q}_k(J) \hookrightarrow C^\infty(W; \mathbb{L})$, where $\mathbb{L}$ is the quantization line bundle; see [**GU, BU**]. (For example, when J is Kähler, by definition, $\mathcal{Q}_k(J) \subset C^\infty(W; \mathbb{L})$.) Thus $\mathcal{V}$ is a subbundle of the trivial bundle $\mathcal{U} \times C^\infty(W; \mathbb{L})$. The connection ∇ comes from the flat connection on the trivial bundle. Namely, let s be a section of $\mathcal{V}$ and let $J(t)$ be a path in $\mathcal{U}$. Denote by $P\colon C^\infty(W; \mathbb{L}) \to \mathcal{Q}_k(J(0))$ the orthogonal projection. Then we set

$$(\nabla_{\dot{J}(0)} s)(J(0)) = P\left(\left.\frac{ds(J(t))}{dt}\right|_{t=0}\right).$$

It would be extremely interesting, already for $W = S^2$, to find an explicit formula for the curvature of this connection and to see how the curvature behaves as $k \to \infty$. As pointed out by Alejandro Uribe, the problem should be accessible using the techniques developed in [**BdMG**]. (See also the results of [**Da**] related to the case where $W = S^2$.)

5.3. Quantization and cobordism. In this section we outline the construction, due to Koprás and Uribe [**KU**], of a homeomorphism between quantization spaces, associated with a cobordism.

Let (M, σ) be a cobordism between symplectic manifolds (W_0, ω_0) and (W_1, ω_1). Assume, for the sake of simplicity, that the orientation of W_0, as a part of ∂M, is symplectic, and the orientation of W_1 is the opposite of symplectic. Let us equip M with a Riemannian metric which is compatible with σ in the natural sense and which is cylindrical near ∂M; see [**KU**]. Such a metric gives rise to a Spin^c-Dirac operator D_M on M which depends on the integer parameter $k > 0$ as σ gets replaced by $k\sigma$. Set $\mathcal{Q}_k(M) = \ker D_M$.

The restriction of $\mathcal{Q}_k(M)$ to ∂M gives rise (in a non-obvious way) to a linear subspace in $\mathcal{Q}_k(\partial M) = \mathcal{Q}_k(W_0) \oplus \mathcal{Q}_k(W_1)$. This subspace is the graph of the operator $U_M\colon \mathcal{Q}_k(W_0) \to \mathcal{Q}_k(W_1)$ which corresponds to the cobordism M.

Let us be more specific. Denote by Λ the linear space comprised of the restrictions to ∂M of solutions of the equation $D_M\psi = 0$. Let $H_\geq$ (respectively, $H_>$) be the space spanned by the eigenvectors of the Spin^c-Dirac operator

$D_{\partial M}$ on ∂M with non-negative (respectively, positive) eigenvalues. The quotient $H_0 = H_{\geq}/H_{>}$ is then naturally isomorphic to the quantization of ∂M. In other words, $H_0 = \mathcal{Q}_k(W_0) \oplus \mathcal{Q}_k(W_1)$. Consider the image Λ_{red} of $\Lambda \cap H_{\geq}$ under the projection $H_{\geq} \to H_0$. The operator $H_0 \to H_0$ given by the orthogonal reflection of H_0 in the space Λ_{red} is unitary and sends $\mathcal{Q}_k(W_0)$ to $\mathcal{Q}_k(W_1)$; see [**KU**]. By definition, the restriction $U_M \colon \mathcal{Q}_k(W_0) \to \mathcal{Q}_k(W_1)$ of this operator corresponds to the cobordism M. Moreover, the semiclassical properties of U_M turn out to be related to the Lagrangian relation associated with M.

This Lagrangian relation is the graph of an (almost-everywhere defined) symplectic mapping, and in this case U_M can be viewed as a quantization of this symplectic mapping. (An interesting example, described by Koprás and Uribe in their paper, is a symplectic cobordism between two 2-dimensional tori for which the symplectic map is the classical "Baker's transformation.")

The analytical aspects, entirely omitted here, of the construction we have just outlined are quite non-trivial. The reader interested in details and examples should consult [**KU**].

Remark H.17. The results of this section extend to the equivariant setting in a straightforward way: when the manifolds are equipped with Hamiltonian actions of a Lie group G, quantization spaces become representations of G and the operators corresponding to cobordisms commute with the G-action.