APPENDIX G

Non-degenerate abstract moment maps

A moment map Ψ associated with a symplectic form satisfies a certain non-degeneracy condition. For example, for a circle action, the Hessian $d^2\Psi$ is non-degenerate on the normal bundle to the fixed point set. In this appendix we will show that a condition of this type is also sufficient in order for Ψ to be locally associated with a symplectic form. We will also derive some topological consequences of this non-degeneracy requirement similar to the properties of moment maps for Hamiltonian torus actions on symplectic manifolds. Implicitly, non-degenerate abstract moment maps have already been used to obtain topological results of this type; see e.g., [**LT2**].

1. Definitions and basic examples

Let G be a torus acting on a manifold M.

Definition G.1. An abstract moment map $\Psi\colon M \to \mathfrak{g}^*$ is *non-degenerate* if for every vector $\xi \in \mathfrak{g}$,

(1) $\mathrm{Crit}(\Psi^\xi) = \{\xi_M = 0\}$, and
(2) $\Psi^\xi\colon M \to \mathbb{R}$ is a Morse-Bott function.

Remark G.2. Condition G.1 is equivalent to:

(1′) for all $p \in M$, the image $d\Psi_p(T_pM)$ is equal to the annihilator $\mathfrak{g}_p^\perp \subseteq \mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}_p$ of the stabilizer of p.

Given that this condition is satisfied, Condition G.1 is equivalent to

(2′) For every subtorus H of G and every $p \in M^H$ the subspace of T_pM annihilated by the quadratic forms

$$d^2\Psi_p^\xi, \quad \text{where } \xi \in \mathfrak{h}, \tag{G.1}$$

is the space T_pM^H.

Remark G.3. The Hessians $d^2\Psi_p^\xi$ are well defined, since, by assumption, $d\Psi_p^\xi = 0$ when $\xi \in \mathfrak{h}$. Also notice that for *any* abstract moment map $\operatorname{im} d\Psi_p \subset \mathfrak{g}_p^\perp$ and T_pM^H is contained in the common annihilator space of the quadratic forms (G.1). Thus Ψ is non-degenerate if and only if these inclusions are equalities.

Remark G.4. Note that for an abstract moment map $\Psi\colon M \to \mathfrak{g}^*$ it makes sense to assert that Ψ is non-degenerate *at a point* $p \in M$. The set of points where Ψ is non-degenerate is open.

Example G.5. Let $\Psi\colon M \to \mathfrak{g}^*$ is a non-degenerate moment map on a G-manifold. Then for any closed subgroup $H \subseteq G$, the H-component $\Psi^H\colon M \to \mathfrak{h}^*$ is a non-degenerate moment map for the H-action on M.

Example G.6. An abstract moment map for a circle action, $\Psi\colon M \to \mathbb{R}$, is non-degenerate if and only if it is a Morse–Bott function and its critical points are the fixed points. In particular, this shows that the manifold M need not have even dimension. However, if the fixed points of the action are isolated (and G is a torus) $\dim M$ is necessarily even.

The basic example of a non-degenerate moment map is still the one associated with a Hamiltonian action of G on a symplectic manifold. In Section 3 we will discuss the converse: given a non-degenerate abstract moment map Ψ, when is Ψ associated to a symplectic form? In particular, we will give conditions under which Ψ is always locally associated with a symplectic form if the dimension of M is even.

Example G.7. Let M be a G manifold. Consider the product $N \times M$, where G acts on N trivially. Let $\Psi\colon N \times M \to \mathfrak{g}^*$ be an abstract moment map. Suppose that Ψ is independent of the N component, i.e., is a pull-back of $\varphi\colon M \to \mathfrak{g}^*$. Then φ is an abstract moment map, and Ψ is non-degenerate if and only if φ is non-degenerate.

This example shows that the local existence of symplectic forms cannot be generalized to a global result. This cylinder construction also gives a local model (with N being an interval) for a non-degenerate abstract moment map on an orientable odd–dimensional manifold (see Section 3.2).

Remark G.8. Definition G.1 makes sense even when G is a non-commutative Lie group. However, in this case it is not clear whether Definition G.1 is sufficient to guarantee any of the topological consequences of non-degeneracy or that an abstract moment map is locally associated with a symplectic form (cf. Section 3 of Appendix E).

2. Global properties of non-degenerate abstract moment maps

Many subtle properties of symplectic manifolds with Hamiltonian torus actions are, in fact, formal consequences of the non-degeneracy of their moment maps. As a result, these properties are also shared by manifolds with non-degenerate *abstract* moment maps. Below we state four theorems establishing such properties.

Let M be a compact manifold acted upon by a torus G and let Ψ be a non-degenerate abstract moment map for this action.

The first two theorems (Theorems G.9 and G.13) are due to Kirwan [**Kir**] in the case of Hamiltonian actions of arbitrary compact Lie groups. The proof of Theorem G.13 given below seems to be new.

2.1. Formality.

Theorem G.9 (Formality). *Let a torus G act on a compact manifold M, and suppose that there exists a non-degenerate abstract moment map, $\Psi\colon M \to \mathfrak{g}^*$. Then*

(1) *the restriction homomorphism*
$$j^*\colon H^*_G(M) \to H^*_G(M^G) = H^*(M^G) \otimes H^*(BG)$$
is a monomorphism,
(2) *the G-manifold M is formal; in particular, $H^*_G(M) \cong H^*(M) \otimes H^*(BG)$ as $H^*(BG)$-modules, and*
(3) *the forgetful homomorphism $H^*_G(M) \to H^*(M)$ is an epimorphism.*

REMARK G.10. Recall that formality means that the Serre spectral sequence converging to the equivariant cohomology of M collapses at the E_2-term and $H^*(M) \otimes H^*(BG)$ is the graded $H^*(BG)$-algebra associated with a certain filtration on $H^*_G(M)$. In particular, the $H^*(BG)$-module structure on $H^*_G(M)$ does not see the G-action; the action appears "trivial" to it. (See Section 4 of Appendix C.)

PROOF OF THEOREM G.9. Let us prove the second assertion first. The non-degeneracy of Ψ implies that for $\xi \in \mathfrak{g}$, the function $f = \langle \Psi, \xi \rangle$ is a Morse–Bott function on M. Thus $\dim H^*(\operatorname{Crit} f) \geq \dim H^*(M)$. On the other hand, $\operatorname{Crit} f = M^G$ for a generic ξ, and so

$$\dim H^*(M^G) \geq \dim H^*(M). \tag{G.2}$$

Due to the non-degeneracy of Ψ, the critical points of f have even indices when these points, i.e., the fixed points of G, are isolated. Thus, in this case f is a perfect Morse function and $\dim H^*(\operatorname{Crit} f) = \dim H^*(M)$. We will show that this is also true in general and thus f is a perfect Morse–Bott function.)

The Serre spectral sequence of the fibration $EG \times_G M \to BG$ converges to $H^*_G(M)$ and has $E_2 = H^*(M) \otimes \mathcal{R}_G$, where $\mathcal{R}_G = H^*(BG)$. This implies

$$\operatorname{rk}_{\mathcal{R}_G} E_2 \geq \operatorname{rk}_{\mathcal{R}_G} H^*_G(M). \tag{G.3}$$

Equality holds if and only if the spectral sequence collapses, for the E_2 term has no $\mathcal{R}_G$-torsion (cf. Section 4 of Appendix C). By equation (C.9), which is a consequence of Borel's localization theorem, $\operatorname{rk}_{\mathcal{R}_G} H^*_G(M) = \dim H^*(M^G)$. (See Theorem C.20 or [**Bor2**] and [**AB2**].) Combining this fact with (G.2) and (G.3), we obtain

$$\operatorname{rk}_{\mathcal{R}_G} H^*_G(M) = \dim H^*(M^G) \geq \dim H^*(M) = \operatorname{rk}_{\mathcal{R}_G} E_2 \geq \operatorname{rk}_{\mathcal{R}_G} H^*_G(M).$$

Therefore, all inequalities in this series are in fact equalities, and the spectral sequence collapses. Thus $E_2 = H^*_G(M)$ as modules over $\mathcal{R}_G$. This proves the second assertion.

In particular, the forgetful homomorphism is onto and $H^*_G(M)$ is a free $\mathcal{R}_G$-module. Again by Borel's localization theorem, j^* is an isomorphism modulo $\mathcal{R}_G$-torsion. Since $H^*_G(M)$ is torsion-free, j^* is a monomorphism. [1] □

The proof of Theorem G.9 only used one, generic, component of the abstract moment map. In fact, the proof shows that if a G-manifold admits a Morse-Bott function whose critical set coincides with the fixed point set, the manifold is a formal G-space. In what follows we will need a relative version of this:

LEMMA G.11. *Let N be a G-manifold with boundary such that the G-action on the boundary has no fixed points. Assume that there exists a function $\varphi\colon N \to \mathbb{R}$ with the following properties:*

(1) *φ is Morse-Bott,*
(2) $\operatorname{Crit} \varphi = N^G$, *and*
(3) $\varphi^{-1}(0) = \partial N$.

Then the pair $(N, \partial N)$ is formal.

[1] It is important to note that in this proof we do not directly use the fact that the connected components of $\operatorname{Crit} f$ have even Morse indices (nor that f is G-invariant). The only properties of f we need are that $\operatorname{Crit} f = M^G$ and that f is Morse–Bott.

PROOF. By the Serre spectral sequence,

$$\mathrm{rk}_{\mathcal{R}_G} H^*_G(N, \partial N) \leq \mathrm{rk}_{\mathcal{R}_G} E_2 = \dim H^*(N, \partial N). \tag{G.4}$$

By Morse theory,

$$\dim H^*(N, \partial N) \leq \dim H^*(\mathrm{Crit}\, \varphi) = \dim H^*(N^G). \tag{G.5}$$

Finally,

$$\dim H^*(N^G) = \mathrm{rk} H^*_G(N^G) = \mathrm{rk}_{\mathcal{R}} H^*_G(N, \partial N)$$

by Borel localization. This together with (G.5) provides the inequality opposite of (G.4). Therefore, (G.4) is an equality. Thus the spectral sequence collapses, and the pair $(N, \partial N)$ is formal. □

REMARK G.12. In Appendix E we discussed the existence problem for abstract moment maps making no non-degeneracy requirements. The existence problem for non-degenerate abstract moment maps is also of interest. The local existence is an immediate consequence of the local linearization theorem for compact group actions (Theorem B.26 in Appendix B).

The global existence problem for non-degenerate abstract moment maps is much harder already for a compact manifold. Theorem G.9 shows that the existence of a non-degenerate abstract moment map yields a strong constraint on the equivariant cohomology of the manifold. In particular, there are compact manifolds acted on by the circle G which do not admit non-degenerate abstract moment maps. (By Example G.6, on such a manifold M there is no invariant Morse–Bott function Ψ with $\mathrm{Crit}\, \Psi = M^G$.)

It would be interesting to see if formality of the manifold (as in Theorem G.9 and Remark G.10) is also sufficient for the existence of a non-degenerate abstract moment map.

2.2. Kirwan's epimorphism and convexity.

THEOREM G.13 (Kirwan's epimorphism). *Let a torus G act on a compact manifold M, let $\Psi\colon M \to \mathfrak{g}^*$ be a non-degenerate abstract moment map, let λ be a regular value of Ψ and $M_\lambda = \Psi^{-1}(\lambda)$. Then the restriction homomorphism*

$$H^*_G(M) \to H^*_G(M_\lambda) = H^*(M_\lambda/G)$$

is an epimorphism.

Recall that, as in the symplectic case, for a regular value λ the G-action on M_λ is locally free, which implies the identification $H^*_G(M_\lambda) = H^*_G(M_\lambda/G)$.

REMARK G.14. The idea of the proof is to argue inductively by performing "reduction in stages". Namely, let ξ be a generic rational vector in $\mathfrak{g}$. Denote by K the circle generated by ξ in G. The theorem is not hard to prove (cf. Lemma C.30) for circle actions by applying a version of the formality theorem (Theorem G.9) for manifolds with boundary. This version of formality theorem can be proved similarly to Theorem G.9. Thus

$$H^*_K(M) \to H^*_K(\{\Psi^\xi = 0\}) = H^*(\{\Psi^\xi = 0\}/K)$$

is an epimorphism. Moreover, by using an *equivariant* formality theorem one can show that

$$H^*_G(M) \to H^*_G(\{\Psi^\xi = 0\}) = H^*_{G/K}(\{\Psi^\xi = 0\}/K)$$

is an epimorphism. Now it suffices to repeat this procedure for the G/K-action on $\{\Psi^\xi = 0\}/K$ which also admits a "non-degenerate abstract moment map". The problem with this approach arises because the quotient is in general an orbifold, not a manifold. Therefore, to do this one would have to develop an orbifold version of the theory of non-degenerate abstract moment maps.

It is easier, however, to apply Lemma G.15 below which uses equivariant Morse–Bott functions instead of non-degenerate abstract moment maps. Thus, we can carry the K-action along when restricting to $\{\Psi^\xi = 0\}$ rather than dividing by K. This is the crucial point of the proof that allows us to avoid working with orbifolds.

Proof of Theorem G.13. Without loss of generality, we may assume that $\lambda = 0 \in \mathfrak{g}^*$. First, let us prove the following result.

Lemma G.15. *Let X be a compact manifold without boundary, acted upon by a torus G. Let $\varphi\colon X \to \mathbb{R}$ be a function with the following properties:*

1. *φ is Morse–Bott,*
2. *0 is a regular value of φ,*
3. *$\operatorname{Crit}\varphi = X^G$, and*
4. *G acts without fixed points on $\{\varphi = 0\}$.*

Then the restriction map $H^*_G(X) \to H^*_G(\{\varphi = 0\})$ is onto.

Proof of the lemma. Let $X_+ = \{\varphi \geq 0\}$, $X_- = \{\varphi \leq 0\}$, and $X_0 = \{\varphi = 0\} = \partial X_+ = \partial X_-$. By Lemma G.11, the pairs (X_+, X_0) and (X_-, X_0) are formal. By Lemma C.23, this implies that the pair (X, X_0) is formal. By Lemma C.30, the restriction map is onto. $\square$

Let us return to the proof of the theorem. Without loss of generality, assume that $\lambda = 0$. Pick a basis $\xi_1, \dots, \xi_r$ in $\mathfrak{g}$ with the following properties:

1. for all s, $1 \leq s \leq r$, the origin $0 \in \mathbb{R}^s$ is a regular value of the mapping $(f_1, \dots, f_s)$, where $f_i = \langle \Psi, \xi_i \rangle$, $i = 1, \dots, r$.
2. for each ξ_s, the zero set of the corresponding vector field coincides with the G-fixed point set: $M^G = \{(\xi_s)_M = 0\}$.

Let $Y_s = (f_1, \dots, f_s)^{-1}(0)$ for $s = 1, \dots, r$ and $Y_0 = M$. Then Y_s is a G-manifold, and the restriction of Ψ to Y_s is an abstract moment map, which may fail to be non-degenerate. However, for any ξ which is not in the linear span of $\xi_1, \dots, \xi_s$, the component $\Psi^\xi|_{Y_s}$ satisfies Conditions 1 and 2 Definition G.1. In particular, for $\xi = \xi_{s+1}$, the function $f_{s+1}|_{Y_s}\colon Y_s \to \mathbb{R}$ is Morse–Bott, and its critical set is Y_s^G. So this function satisfies Conditions 1–3 of Lemma G.15, with $X = Y_s$ and $\varphi = f_{s+1}|_{Y_s}$. Condition 4 of the lemma follows from Condition 2 because φ is a component of an abstract moment map. By applying this lemma to $s = 0, 1, \dots, r-1$, we obtain a sequence of epimorphisms:

$$H^*_G(M) = H^*_G(Y_0) \twoheadrightarrow H^*_G(Y_1) \twoheadrightarrow \dots \twoheadrightarrow H^*_G(Y_r) = H^*_G(M_0).$$

This completes the proof of the theorem. $\square$

Remark G.16. Even in the symplectic case, $|\Psi|^2$ need not be a Morse-Bott function. However, $|\Psi|^2$ satisfies Kirwan's condition which is weaker than being Morse–Bott but is still sufficient for Morse theory, [**Kir**]. As S. Tolman pointed out, it would be interesting to check whether or not the same is true for non-degenerate abstract moment maps.

THEOREM G.17 (Convexity). *The image $\Psi(M)$ is a convex polytope in $\mathfrak{g}^*$ and for every $\lambda \in \mathfrak{g}^*$ the preimage $M_\lambda = \Psi^{-1}(\lambda)$ is connected.*

The local convexity is an immediate consequence of the assumption that Ψ is non-degenerate. The global convexity follows from the local convexity in the same way as in the symplectic case. (See [**At3**] and [**GS2**].)

REMARK G.18. We emphasize that Condition 1 of Definition G.1 is crucial for Theorem G.17 to hold. The reason is that in order for critical manifolds to have even indices, the G-action on the critical set should be trivial, forcing nearby orbits to be close to particular points of the critical manifold rather than to the entire critical manifold. For example, contact moment maps for invariant contact forms (see [**Ler2**] and also [**Ar**] for the notion of a contact Hamiltonian) may fail to be non-degenerate abstract moment maps in our sense (see Example G.29). Furthermore, Theorem G.17, as stated above, fails in general for such moment maps. (However, a different form of the convexity theorem still holds; see [**Ler2**] and references therein.) This also shows that although Definition G.1 appears completely adequate for working with even–dimensional manifolds, its usefulness in odd-dimension is unclear. We will return to the discussion of the odd–dimensional case in Section 3.2.

Finally, by Theorem G.9, the theorems of Goretsky, Kottwitz and MacPherson, [**GKM**], and the Brion–Vergne theorem, [**BrV**], readily apply to non-degenerate moment maps. Let N be the set of points $p \in M$ such that the stabilizer G_p is one-dimensional and let $\bar{N}$ be the closure of N, i.e., $\bar{N} = N \cup M^G$. As in the symplectic case, the images of the restrictions $H^*_G(M) \to H^*_G(M^G)$ and $H^*_G(\bar{N}) \to H^*_G(M^G)$ coincide:

THEOREM G.19. *The restriction $H^*_G(M) \to H^*_G(\bar{N})$ is an isomorphism.*

Simple proofs of this result, not using the Goretsky–Kottwitz–MacPherson theorem, can be found in [**TW**] and [**BrV**].

3. Existence of non-degenerate two-forms

3.1. Local existence of symplectic forms on even dimensional manifolds.

EXAMPLE G.20. Let $\Psi\colon M \to \mathfrak{g}^*$ be a moment map for a Hamiltonian torus action on a symplectic manifold. Then Ψ is a non-degenerate abstract moment map.

The converse does not hold literally:

EXAMPLE G.21. Let $S^{2n} \subset \mathbb{C}^n \times \mathbb{R}$ be the unit sphere and consider the projection to the $\mathbb{R}$ component,

$$\Psi\colon S^{2n} \to \mathbb{R}.$$

This is a non-degenerate abstract moment map with respect to the S^1 action induced by the standard diagonal S^1 action on $\mathbb{C}^n$. However, by Corollary E.40, it is not even associated with a closed two-form when $2n \geq 4$.

Considering direct products as in Example G.7, it is easy to construct examples of non-degenerate abstract moment maps which do satisfy the hypotheses of Corollary E.40 and are therefore associated with closed two-forms but are not associated with *symplectic* forms. However, we note that we are not aware of such examples in which the fixed points are *isolated*.

In the above examples, the obstruction to the existence of a symplectic form is *global*. However, the non-degeneracy condition for an abstract moment map is *local*. We will show that a non-degenerate abstract moment map is always associated with a symplectic form *locally*, if (and only if) it satisfies an additional condition, which we now derive.

Suppose that $\Psi\colon M \to \mathfrak{g}^*$ is associated with a symplectic form on a neighborhood of an orbit $\mathcal{O} = G \cdot p$. The *symplectic slice* to the orbit at p is

$$S = (T_p\mathcal{O})^\omega / T_p\mathcal{O} \tag{G.6}$$

where $(T_p\mathcal{O})^\omega$ is the "orthogonal complement" to $T_p\mathcal{O}$ in T_pM with respect to ω. The vector space S acquires from M a symplectic form and a linear symplectic G_p action, where G_p is the stabilizer of p in G.

Let H be the connected component of the identity in G_p. Let $K \subseteq G$ be a complementary torus to H in G, i.e., a torus with the property that the multiplication homomorphism

$$K \times H \to G$$

is an isomorphism. Then

$$\Gamma = K \cap G_p \cong G_p/H$$

is a finite abelian group, and we have an isomorphism

$$K \times_\Gamma G_p \cong G \tag{G.7}$$

given by $[k, a] \mapsto k \cdot a$.

Consider the cotangent bundle $T^*K = \mathfrak{k}^* \times K$ with its standard symplectic form, K action, and moment map

$$\mathfrak{k}^* \times K \to \mathfrak{k}^*.$$

Combining this with the symplectic slice, we get an induced symplectic form, G action (through the isomorphism (G.7)), and moment map on the "model"

$$Y := T^*K \times_\Gamma S.$$

The *local normal form* for a Hamiltonian torus action asserts that a neighborhood of $\mathcal{O}$ in M is isomorphic to a neighborhood of the zero section in the model Y (thought of as a bundle over $K/\Gamma \cong G/G_p \cong \mathcal{O}$).

Given a non-degenerate abstract moment map, we establish the local existence of a symplectic form by more or less reversing the above argument. First note that, because $(T_p\mathcal{O})^\omega = \ker d\Phi|_p$, we can express the symplectic slice (G.6) as

$$S = \ker d\Phi|_p / T_p\mathcal{O}.$$

In this form, the slice is defined for *any* non-degenerate abstract moment map Φ. It is then a real vector space with a linear G_p action. A necessary condition for Φ to be associated with a symplectic form near $\mathcal{O}$ is for S to admit a G_p-invariant symplectic form. Because every compact subgroup of the symplectic group $\mathrm{Sp}(2n)$ is conjugate to a subgroup of $U(n)$, we can then identify S with $\mathbb{C}^n$ such that G_p acts by unitary transformations. Therefore, a necessary condition for Φ to be associated with a symplectic form near $\mathcal{O}$ is

(G.8) The slice S admits a G_p-invariant complex structure.

This condition is also sufficient:

THEOREM G.22. *Let $\Psi\colon M \to \mathfrak{g}^*$ be a non-degenerate abstract moment map on an even-dimensional G-manifold, let $\mathcal{O} = G \cdot p \cong G/G_p$ be the orbit of a point $p \in M$, and let $G_p \subseteq G$ be its stabilizer. Suppose that the slice $S = \ker d\Phi|_p / T_p\mathcal{O}$ admits a G_p invariant complex structure. Then there exists an invariant neighborhood of $\mathcal{O}$ on which Ψ is associated with a symplectic form.*

The condition of Theorem G.22 is automatically satisfied in the presence of a stable complex structure:

COROLLARY G.23. Let M be an even-dimensional G-equivariant stable complex manifold, for G a torus. Let $\Psi\colon M \to \mathfrak{g}^*$ be a non-degenerate abstract moment map. Then every G-orbit $\mathcal{O}$ has a neighborhood on which Ψ is associated with a symplectic form.

We will prove Theorem G.22 and Corollary G.23 later in this section. At this point let us just mention that, in Corollary G.23, the given stable complex structure need not be induced from an almost complex structure compatible with the symplectic form.

To begin, we note that on a neighborhood of an orbit $\mathcal{O}$ there always exists a closed two-form for which Ψ is a moment map by Corollary E.33. We need to show that such a two-form can be chosen to be symplectic. Let us begin with the case that p is an isolated fixed point.

LEMMA G.24. *Let M be a G-manifold, for G a torus, and let $p \in M^G$ be an isolated fixed point. Let $\Psi\colon M \to \mathfrak{g}^*$ be an abstract moment map. Let ω be a closed two-form on a neighborhood of p for which Ψ is a moment map. Then ω is non-degenerate at p if and only if Ψ is non-degenerate at p.*

PROOF. From the local normal form for a compact group action we may assume that M is a vector space with a linear G action and $p = 0$ is the only fixed point. Let

$$V_1 \oplus \ldots \oplus V_r$$

be the isotypic decomposition of this vector space. Then there exist weights $\alpha_i \in \mathbb{Z}_G^*$ such that $\alpha_i \neq \alpha_j$ for all $i \neq j$ and V_i is a complex vector space on which G acts with weight α_i, i.e., acts by the character $G \to S^1$, $\exp(\xi) \mapsto e^{\sqrt{-1}\alpha_i(\xi)}$, followed by the standard S^1 action by scalar multiplication.

By G-invariance, it is not hard to show that

$$\omega|_p = \omega_1 \oplus \ldots \oplus \omega_r$$

where ω_i is an S^1-invariant two-form on V_i.

Let us first assume that $\dim_{\mathbb{C}} V_i = 1$ for all i. Then there exist $c_1, \ldots, c_r$ such that

$$\omega|_p = -\sqrt{-1} \sum c_j dz_j d\overline{z}_j \tag{G.9}$$

where z_j is a complex coordinate on V_j. The action is generated by the vector fields

$$\xi_M = \sqrt{-1} \sum \alpha_j(\xi) \left(z_j \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \right).$$

This implies that

$$d^2\Psi^\xi(p) = \sum c_j \alpha_j(\xi) |z_j|^2.$$

Therefore, the abstract moment map Ψ is non-degenerate at p exactly if all the c_j's are non-zero, which holds exactly if the two-form (G.9) is non-degenerate at p.

Let us now allow the V_i to be of any dimension.

LEMMA G.25. *Let ω be an S^1 invariant alternating two-form on a complex vector space V. Then there exist coordinates z_j, for $1 \leq j \leq \dim V$, such that*

$$\omega = \sum c_j dz_j \wedge d\overline{z}_j$$

for some constant c_j.

PROOF OF THE LEMMA. By S^1 invariance, the kernel of ω is a complex subspace of V. It is enough to bring ω to the desired form on a complementary complex subspace W, on which ω is non-degenerate. By S^1-invariance,

$$\omega = \sum c_{ij} dz_i \wedge d\overline{z}_j$$

where z_i are complex coordinates on W and $c_{ji} = -\overline{c}_{ij}$. There exists a one-dimensional complex subspace W_1 on which ω is non-degenerate. Indeed, if $c_{ij} \neq 0$, we can take W_1 to be generated by the real part of the vector $\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \overline{z}_j} \in W \otimes_{\mathbb{R}} \mathbb{C}$. Because ω is S^1 invariant, the "orthogonal complement" of W_1 with respect to ω is a complex vector space. By induction, we get $W = W_1 \oplus \ldots \oplus W_m$ where each W_i is a one-dimensional complex vector space and $\omega|_{W_i \times W_j} = 0$ if $i \neq j$. The lemma follows. □

We can now finish the proof of Lemma G.24. Applying Lemma G.25 to each of the spaces V_i, we find coordinates z_j, for $1 \leq j \leq \dim M$ on $V_1 \oplus \ldots \oplus V_r$ such that $\omega|_p = \sum c_j dz_j \wedge d\overline{z}_j$ and $d^2\Psi^\xi(p) = \sum c_j \alpha_j(\xi)|z_j|^2$. As before, ω is non-degenerate at p if and only if $c_j \neq 0$ for all j, if and only if Ψ is non-degenerate at p. □

Lemma G.24 does not hold when G is disconnected. For example, the zero map is a non-degenerate abstract moment map for a Γ action if Γ is finite but the zero two-form is not non-degenerate. Moreover, in this situation there might not even *exist* an invariant symplectic form:

EXAMPLE G.26. Let Γ denote the group of transformations of $\mathbb{R}^{2n}$ of the form

$$(x_1, \ldots, x_{2n}) \mapsto (\epsilon_1 x_1, \ldots, \epsilon_{2n} x_{2n})$$

with $\epsilon_i \in \{-1, 1\}$ for each i and $\prod \epsilon_i = 1$. If $2n > 2$, the only Γ-invariant two-form in $\bigwedge^2(\mathbb{R}^n)^*$ is zero. (This can be see easily by writing an arbitrary two-form in coordinates as $\sum_{i<j} a_{ij} dx_i \wedge dx_j$.)

Our assumption that the slice S admits an invariant complex structure allows us to avoid situations as in Example G.26. We will need the following technical result:

LEMMA G.27. *Let a compact abelian Lie group G act on an even-dimensional vector space S. Suppose that there exists a G-invariant complex structure on S. Consider the product $N \times S$ where N is a manifold with a trivial G-action, and the point $(\alpha, 0)$, where $\alpha \in N$. Let*

$$\Psi \colon N \times S \to \mathfrak{g}^*$$

be an abstract moment map which is non-degenerate at $(\alpha, 0)$. Then there exists a closed two-form on $N \times S$ for which Ψ is a moment map and whose restriction to $\{\alpha\} \times S$ is symplectic at $(\alpha, 0)$.

Proof. By Corollary E.33, there exists a closed two-form σ_1 on $N \times S$ for which Ψ is a moment map. Let $H \subset G$ be the component of the identity element. Let S^H be the subspace of S fixed by H. Then there exists a G-invariant complementary subspace S' such that $S = S^H \oplus S'$, so that

$$N \times S = N \times S^H \times S'.$$

It is not hard to see that the restriction of Ψ to $\{\alpha\} \times \{0\} \times S'$ is non-degenerate at $(\alpha, 0, 0)$. By Lemma G.24, the restriction of σ_1 to $\{\alpha\} \times \{0\} \times S'$ is symplectic. Consider the (non-effective) G action on $S^H \cong \mathbb{C}^m$. Because every compact subgroup of $\mathrm{GL}(m, \mathbb{C})$ is conjugate to a subgroup of $\mathrm{U}(m)$, there exists a G-invariant symplectic form on S^H. Let σ_2 be its pull-back to $N \times S^H \times S'$. Then for a large enough r, the restriction of $\omega := \sigma_1 + r\sigma_2$ to $\{\alpha\} \times S$ is symplectic at $(\alpha, 0)$, and Ψ is a moment map for ω. □

Proof of Theorem G.22. As before, let us write

$$G = K \times_\Gamma G_p \tag{G.10}$$

with Γ finite abelian. Because K acts with a finite stabilizer at p, and by the first condition of a non-degenerate abstract moment map, the K-component of Ψ is a submersion at p, i.e.,

$$d\Psi^K|_p \colon T_pM \to \mathfrak{k}^*$$

is onto. Because all points of $\mathcal{O}$ have the same stabilizer, this is true at all points of $\mathcal{O}$. Let $\alpha = \Psi^K(p) \in \mathfrak{k}^*$. Then there exists an invariant neighborhood Z of $\mathcal{O}$ in $(\Psi^K)^{-1}(\alpha)$ which is a manifold and an equivariant diffeomorphism between a neighborhood of $\mathcal{O}$ in M and a neighborhood of $\{\alpha\} \times \mathcal{O}$ in $\mathfrak{k}^* \times Z$ which carries Ψ^K to the projection map, which we denote by the same symbol:

$$\Psi^K \colon \mathfrak{k}^* \times Z \to \mathfrak{k}^*.$$

A neighborhood of $\mathcal{O}$ in Z is equivariantly diffeomorphic to a neighborhood of $G \times_{G_p} \{0\}$ in

$$G \times_{G_p} S, \tag{G.11}$$

where $S = T_pZ/T_p\mathcal{O} = \ker d\Psi|_p/T_p\mathcal{O}$ is the slice; this follows from the local normal form for a compact group action (see Appendix B). We rewrite (G.11) as

$$K \times_\Gamma S,$$

where G acts through the decomposition (G.10), the K action on itself, and the G_p action on S. Therefore, we may assume that

$$M = k^* \times K \times_\Gamma S$$

and $p = [\alpha, e, 0]$. The decomposition (G.10) gives a decomposition $\mathfrak{g}^* = \mathfrak{k}^* \oplus \mathfrak{g}_p^*$ so that

$$\Psi = \Psi^K + \Psi^{G_p}.$$

Recall that $\Psi^K \colon M \to \mathfrak{k}^*$ is the projection to the first coordinate.

By K-invariance, Ψ^{G_p} is the pull-back to M of a non-degenerate abstract moment map, which we denote by the same symbol,

$$\Psi^{G_p} \colon \mathfrak{k}^* \times S \to \mathfrak{g}_p^*.$$

Let ω_1 be the pull-back to M of the standard symplectic form on the cotangent bundle $T^*K = \mathfrak{k}^* \times K$. Note that its G_p moment map is zero.

By Lemma G.27, there exists a closed two-form on $\mathfrak{k}^* \times S$ for which Ψ^{G_p} is a moment map and whose restriction to $\{\alpha\} \times S$ is symplectic at $(\alpha, 0)$; let ω_2 be its pull-back to M. Note that its K moment map is zero.

Then $\omega_1 + \omega_2$ is a closed two-form for which G $\Psi^K + \Psi^{G_p}$ is a moment map and which is symplectic at p, and hence on an entire neighborhood of $\mathcal{O}$. □

Proof of Corollary G.23. By Theorem G.22 it is enough to show that for each $p \in M$ the slice $S = \ker d\Phi|_p / T_p\mathcal{O}$ admits a G_p-invariant complex structure. As before, we decompose our torus as $G = K \times_\Gamma G_p$ and the moment map as $\Psi = \Psi^K \oplus \Psi^{G_p}$. Then $\ker d\Psi|_p = \ker d\Psi^K|_p$, and $T_pM \cong \ker d\Psi^K|_p \oplus \mathfrak{k}^*$, where G_p acts on $\mathfrak{k}^*$ trivially. Also, $\ker d\Psi|_p \cong S \oplus T_p\mathcal{O}$, where G_p acts on $T_p\mathcal{O}$ trivially. An equivariant stable complex structure on M then gives a G_p-invariant complex structure on $T_pM \oplus \mathbb{R}^m = S \oplus T_p\mathcal{O} \oplus \mathfrak{k}^* \oplus \mathbb{R}^m$ for some m. Because the subspace fixed by G_p is $(T_pM \oplus \mathbb{R}^m)^{G_p} = S^{G_p} \oplus T_p\mathcal{O} \oplus \mathfrak{k}^* \oplus \mathbb{R}^m$, this is a complex subspace, and the quotient

$$(T_pM \oplus \mathbb{R}^m)/(T_pM \oplus \mathbb{R}^m)^G \cong S/S^{G_p}$$

acquires a G_p-invariant complex structure. Finally, because $S \cong S/S^{G_p} \oplus S^{G_p}$, and because S^{G_p} is an even-dimensional vector space with a trivial G_p action, an arbitrary complex structure on S^{G_p} and the G_p-invariant complex structure on S/S^{G_p} fit together to give a G_p invariant complex structure on S. □

3.2. Odd-dimensional analogues. In this section we briefly touch upon the question of the local structure of non-degenerate moment maps on odd–dimensional manifolds.

Example G.28. Let ω be a closed G-invariant two-form of maximal rank on an odd–dimensional manifold M such that the field of directions $\ker \omega$ (called the characteristic field) is nowhere tangent to G-orbits. Also, let Ψ be an abstract moment map associated with ω. (Similarly to the the symplectic case, Ψ always exists locally under the above non-tangency condition, but need not exists globally.) Then Ψ is non-degenerate.

Example G.29. In particular we note, continuing Remark G.18, that a contact moment map is non-degenerate exactly at the points where the Reeb vector field is not tangent to the G-orbit. (For a compact contact manifold, a tangency always occurs at some points.)

In fact, the converse of Example G.28, i.e., an odd–dimensional analogue of Theorem G.22, also holds locally. First note that arguing as in Section 3.1, it is not hard to prove the following result.

Theorem G.30. *Assume that M is odd–dimensional and orientable and let Ψ be a non-degenerate abstract moment map on M. Then locally (M, Ψ) is a cylinder over an even–dimensional manifold with abstract moment map. In other words, for every point $p \in M$, there exists a G-invariant open neighborhood U containing p such that U is equivariantly diffeomorphic to the product $U_0 \times I$, where U_0 is even-dimensional and I is an interval with trivial G-action, and Ψ is independent of the I-component.*

Applying Theorem G.22 we conclude that, if the slice in U_0 admits an invariant complex structure, then locally M is a product of a symplectic manifold and an interval, where the action is trivial on the interval and the moment map is independent of the second component.

Alternatively, this can be expressed as the odd-dimensional analogue of Theorem G.22. Namely, for every point $p \in M$, there exists a G-invariant open set U containing p and a G-invariant form ω on U of maximal rank such that the action of G on (U, ω) is Hamiltonian and Ψ is the moment map associated with this action. The characteristic foliation of ω is invariant under the G-action, nowhere tangent to the G-orbits, and Ψ is constant along this foliation.

All of these statements (including Theorem G.30) are equivalent to each other and can be easily derived from Theorem G.22. For example, let us prove the local existence of ω, which readily implies Theorem G.30. First, we apply Theorem G.22 to the product $M \times \mathbb{R}$ where the G-action on $\mathbb{R}$ is trivial and Ψ is extended to $M \times \mathbb{R}$ to be independent of the second component. Then the form ω is obtained by restricting the symplectic form on a neighborhood of $(p, 0)$ to the intersection of this neighborhood with $M \times \{0\}$. It is clear that ω has maximal rank. Since the form ω is invariant, the characteristic field $\ker \omega$ is G-invariant. Since Ψ is associated with ω, it is constant along characteristics. Finally, the first non-degeneracy condition implies that $\ker \omega$ is nowhere tangent to G-orbits.