APPENDIX J

Cobordism invariance of the index of a transversally elliptic operator

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1. The $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator and the $\mathrm{Spin}^{\mathbb{C}}$-quantization

In this section we reformulate the "$\mathrm{Spin}^{\mathbb{C}}$-quantization commutes with cobordism" principal in a more analytic language. In Subsection 1.1, we briefly recall the notion of $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator. We refer the reader to [**BGV, Du**] for details. We also express the $\mathrm{Spin}^{\mathbb{C}}$-quantization of an orbifold $M = X/G$ in terms of the index of the lift of the $\mathrm{Spin}^{\mathbb{C}}$-Dirac operator to X. In Subsection 1.2, we explain the relationship between such lifts associated with cobordant $\mathrm{Spin}^{\mathbb{C}}$-structures. This leads us to a notion of a cobordism between transversally elliptic operators. To show that $\mathrm{Spin}^{\mathbb{C}}$-quantization of orbifolds commutes with cobordism, it is then enough to prove that the index of transversally elliptic operators commutes with cobordisms, which will be shown in the subsequent sections.

1.1. A Dirac operator associated with a $\mathrm{Spin}^{\mathbb{C}}$-structure. Suppose M is a compact oriented m-dimensional orbifold. Let (P, p) be a $\mathrm{Spin}^{\mathbb{C}}$-structure on M. Recall from Section D.2 that here P is a principal $\mathrm{Spin}^{\mathbb{C}}(n)$-bundle over M and $p : P \to \mathrm{GL}_+(TM)$ is a $\mathrm{Spin}^{\mathbb{C}}(n)$-equivariant map, which gives rise to an $\mathrm{SO}(n)$-structure on TM and, thus, to a Riemannian metric g^M on M.

The group $\mathrm{Spin}^{\mathbb{C}}(n)$ has a canonical unitary representation S, called the *space of spinors*. The Hermitian orbibundle $\mathcal{S} = P \times_{\mathrm{Spin}^{\mathbb{C}}(n)} S$ is called the *spinor bundle* over M.

Example. If the $\mathrm{Spin}^{\mathbb{C}}$-structure on M is given by an almost complex structure and a complex line bundle $\mathbb{L}$ (cf. Section D.5), then the spinor bundle is isomorphic to $\Lambda^\bullet(T^{0,1}M)^* \otimes \mathbb{L}$, where $\Lambda^\bullet(T^{0,1}M)^*$ is the bundle of anti-holomorphic forms on M.

There is a canonical action $c : TM \to \mathrm{End}(\mathcal{S})$ of the tangent bundle TM on $\mathcal{S}$ by skew-adjoint endomorphisms, such that $c(v)^2 = -|v|^2$ where $|v|$ denotes the norm of the vector $v \in TM$ with respect to the Riemannian metric g^M.

Let $\nabla^{\mathcal{S}}$ be a Hermitian connection on $\mathcal{S}$. The $\mathrm{Spin}^{\mathbb{C}}$-*Dirac operator* on $\mathcal{S}$ is the first order differential operator

$$D := \sum_{i=1}^{n} c(e_i)\nabla^{\mathcal{S}}_{e_i} : C^\infty(M,\mathcal{S}) \to C^\infty(M,\mathcal{S}), \tag{J.1}$$

where $e_1, \dots, e_m$ is an orthonormal frame of TM (the operator D is independent of the choice of this frame). If the connection $\nabla^{\mathcal{S}}$ was properly chosen, which we will henceforth assume, then the operator D is self-adjoint; cf. [**BGV**, Proposition 3.44].

Suppose now that $m = \dim M$ is even. Then the spinor bundle possesses a natural grading $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ such that

$$D : C^\infty(M, \mathcal{S}^\pm) \ \to \ C^\infty(M, \mathcal{S}^\mp).$$

We denote by $D^\pm$ the restriction of D to the space $C^\infty(M, \mathcal{S}^\pm)$. Note that $(D^+)^* = D^-$.

By definition, the $\mathrm{Spin}^{\mathbb{C}}$-quantization $Q(M)$ of M is equal to the index the operator D^+:

$$Q(M) \ := \ \dim \operatorname{Ker} D^+ \ - \ \dim \operatorname{Coker} D^+ \ = \ \dim \operatorname{Ker} D^+ \ - \ \dim \operatorname{Ker} D^-.$$

Suppose now that $M = X/G$ is a presentation of the orbifold M. Here X is a smooth compact manifold with a locally free action of a compact group G. Let g^X be the lift of the Riemannian metric on M to X. Then the action of G on X preserves g^X.

Denote by $\tilde{D}^\pm$ the lift of $D^\pm$ to X. Then $\tilde{D}^\pm$ is a G-invariant transversally elliptic operator on X (see Subsection 3.1 for a definition of a transversally elliptic operator). The group G acts on the kernel of $\tilde{D}^\pm$ and $\operatorname{Ker} D^\pm$ is naturally isomorphic to the G-invariant part $\left(\operatorname{Ker} \tilde{D}^\pm\right)^G$ of $\operatorname{Ker} \tilde{D}^\pm$; cf. [**Kaw2**, §1]. Hence,

$$Q(M) \ = \ \dim \left(\operatorname{Ker} \tilde{D}^+\right)^G \ - \ \dim \left(\operatorname{Ker} \tilde{D}^-\right)^G.$$

1.2. A cobordism of Dirac operators. Let $M_0 = X_0/G$ and $M_1 = X_1/G$ be presented orbifolds of the same even dimension $m = 2k$, endowed with $\mathrm{Spin}^{\mathbb{C}}$-structures. Assume that there is given a $\mathrm{Spin}^{\mathbb{C}}$-cobordism between M_0 and M_1. In other words, we assume the following data:

A compact oriented manifold W with boundary; a locally free action of a compact Lie group G on W; the representation of the boundary of W as a disjoint union of $-X_0$ and X_1 (here, $-X_0$ is the manifold X_0 with the opposite orientation); the $\mathrm{Spin}^{\mathbb{C}}$-structure on W/G, which, by restriction, induces $\mathrm{Spin}^{\mathbb{C}}$-structures on $X_i/G,\ i = 0, 1$; the isomorphisms between the orbifolds M_i and $X_i/G,\ i = 0, 1$, which carry the $\mathrm{Spin}^{\mathbb{C}}$-structures on M_i to the $\mathrm{Spin}^{\mathbb{C}}$-structures on X_i/G.

Recall from the previous subsection, that there is a natural Riemannian metric on W. Using this metric we can identify the union of the cylinders $X_0 \times [0, \varepsilon)$ and $X_1 \times (-\varepsilon, 0]$ with a neighborhood $U \subset W$ of the boundary of W. We denote by $t : U \to \mathbb{R}$ the projection onto the second factor. By a slight abuse of notation, we denote by the same latter t the induced map $U/G \to \mathbb{R}$. Let $dt \in T(U/G) \subset T(W/G)$ be the corresponding vector (we use the Riemannian metric to identify the tangent and the cotangent bundles to W/G). Set

$$\gamma \ := \ c(dt).$$

Let $\mathcal{S}, \mathcal{S}_i$ denote the spinor bundles on W/G and $M_i = X_i/G$ respectively. Recall from the previous section that there is a natural grading $\mathcal{S}_i = \mathcal{S}_i^+ \oplus \mathcal{S}_i^-$. There is a natural isomorphism between the restriction of $\mathcal{S}$ to M_i and $\mathcal{S}_i$. Under this isomorphism we have $\gamma|_{\mathcal{S}_i^\pm} = \pm\sqrt{-1}$.

Fix a connection on $\mathcal{S}$. It induces connections on $\mathcal{S}_i$. Let $\tilde{D}, \tilde{D}_i$ denote the lifts of the corresponding Dirac operators to W and X_i respectively. From (J.1), we see that the restriction of $\tilde{D}$ to the neighborhood of $X_i,\ i=0,1$ has the form

$$\tilde{D} \;=\; \gamma\frac{\partial}{\partial t} \;+\; \tilde{D}_i, \tag{J.2}$$

where $\frac{\partial}{\partial t}$ denotes the covariant derivatives along t.

Thus to show that the $\mathrm{Spin}^{\mathbb{C}}$-quantizations commute with cobordism, it is enough to prove that the indexes of the transversally elliptic operators $\tilde{D}_i$ on $X_i,\ i=0,1$ coincide, whenever there exists a G-equivariant cobordism W between X_0 and X_1 and a G-invariant transversally elliptic operator $\tilde{D}$ on W, whose restriction to a neighborhood of the boundary satisfies (J.2). This statement will be made more precise and proven below.

2. The summary of the results

2.1. Let X be a compact n-dimensional Riemannian manifold on which a compact Lie group G acts by isometries. Let E^+, E^- be G-equivariant Hermitian vector bundles over X. Let $A^+ : C^\infty(X,E^+) \to C^\infty(X,E^-)$ be a G-invariant transversally elliptic differential operator of order 1 (cf. [**At1**] or Section 3 of this appendix).

Let $A^- : C^\infty(X,E^-) \to C^\infty(X,E^+)$ be the formal adjoint of A^+ and consider the operator

$$A \;:=\; \begin{bmatrix} 0 & A^- \\ A^+ & 0 \end{bmatrix} :\; C^\infty(X, E^+\oplus E^-) \;\to\; C^\infty(X, E^+\oplus E^-).$$

2.2. The distributional index. The kernel $\mathrm{Ker}(A^\pm) \subset L^2(X,E^\pm)$ is a closed G-invariant subspace, and, hence, can be considered as unitary representations of G.

Let us denote by $\hat{G}$ the set of all equivalence classes of irreducible representations of G. For $\rho\in\hat{G}$, we denote by $\mathrm{Ker}_\rho(A^\pm) := \mathrm{Hom}_G(\rho, \mathrm{Ker}(A^\pm))$ the ρ-component of $\mathrm{Ker}(A^\pm)$.

Atiyah, [**At1**], showed that, for each $\rho\in\hat{G}$, the dimension of the spaces $\mathrm{Ker}_\rho(A^\pm)$ is finite. Moreover, the formal sum

$$\mathrm{char}\,\mathrm{Ker}(A^\pm) \;:=\; \sum_{\rho\in\hat{G}} \dim \mathrm{Ker}_\rho(A^\pm)\cdot \mathrm{char}\,\rho \tag{J.3}$$

converges to a distribution on G.

The *distributional index* $\mathrm{ind}^G(A)$ is defined by

$$\mathrm{ind}^G A \;:=\; \mathrm{char}\,\mathrm{Ker}(A^+) \;-\; \mathrm{char}\,\mathrm{Ker}(A^-) \;\in \mathcal{D}'(G)^{\mathrm{inv}}, \tag{J.4}$$

where $\mathcal{D}'(G)^{\mathrm{inv}}$ denotes the space of distributions on G invariant under the inner automorphisms of G. Set $\mathrm{ind}^G_\rho A := \dim\mathrm{Ker}_\rho(A^+) - \dim\mathrm{Ker}_\rho(A^+)$. Then

$$\mathrm{ind}^G A \;=\; \sum_{\rho\in\hat{G}} \mathrm{ind}^G_\rho(A)\cdot\mathrm{char}\,\rho. \tag{J.5}$$

2.3. The case when X is a boundary. Suppose now that X is a boundary of a compact G-manifold W and that F is a G-equivariant vector bundle over W, whose restriction to X is G-equivariantly isomorphic to E. Note, that we do not assume that the bundle F is graded.

We choose an equivariant identification of a neighborhood U of the boundary of W with the product $X \times (-\varepsilon, 0]$ and we denote by $t : X \times (-\varepsilon, 0] \to (-\varepsilon, 0]$ the projection. We fix a G-equivariant connection on F, so that the operator $\partial/\partial t$ acts on the sections of the restriction of F to U.

The main result of this appendix is the following

Theorem 1. *Assume that there exists a self-adjoint G-invariant transversally elliptic symmetric differential operator $B : C^\infty(W, F) \to C^\infty(W, F)$, whose restriction to U has the form*

$$B \;=\; \gamma \frac{\partial}{\partial t} \;+\; A, \tag{J.6}$$

where γ is a skew-adjoint bundle map, such that $\gamma|_{E^\pm} = \pm\sqrt{-1}$. Then the index $\operatorname{ind}^G A \;=\; 0$.

Remark 1. By (J.5), the theorem is equivalent to the statement that $\operatorname{ind}^G_\rho A = 0$ for all $\rho \in \hat{G}$.

2.4. The cobordism invariance. Theorem 1 implies the cobordism invariance of the index in the following sense.

Assume that $X_i, i = 0, 1$ are compact Riemannian G-manifolds and that $E_i^\pm$ are G-equivariant Hermitian vector bundles over X_i. Let $A_i^+ : C^\infty(X_i, E_i^+) \to C^\infty(X_i, E_i^-)$ be transversally elliptic differential operators. Set $A_i = A_i^+ \oplus A_i^-$.

Suppose W is a compact G-manifold, whose boundary is the disjoint union of X_0 and X_1. Then we can and will identify a neighborhood U of the boundary of W with the disjoint union of the cylinders $X_0 \times [0, \varepsilon)$ and $X_1 \times (-\varepsilon, 0]$. We denote by $t : U \to \mathbb{R}$ the projection onto the second factor.

Assume that there exist a G-equivariant Hermitian vector bundle F over W, whose restriction to the boundary is isomorphic to the bundle induced by $E_i = E_i^+ \oplus E_i^-$, and an operator $B : C^\infty(W, F) \to C^\infty(W, F)$, which near the boundary takes the form (J.6). In this situation we say that the operators A_0 and A_1 are *cobordant*.

Let $E_0^{\mathrm{op}} = E_0^{\mathrm{op}+} \oplus E_0^{\mathrm{op}-}$ be the bundle E_0 with the opposite grading, i.e., $E_0^{\mathrm{op}\pm} = E_0^{\mp}$. Then A_0 defines the operator $A_0^{\mathrm{op}} : C^\infty(X_0, E_0^{\mathrm{op}}) \to C^\infty(X_i, E_0^{\mathrm{op}})$. Clearly, $\operatorname{ind}^G A_0^{\mathrm{op}} = -\operatorname{ind}^G A_0$.

Set $X = X_0 \sqcup X_1$ and let $E = E^+ \oplus E^-$ be the graded bundle over X induced by E_0^{op} and E_1. Let $A : C^\infty(X, E) \to C^\infty(X, E)$ be the operator induced by A_0^{op} and A_1. Then the operators A and B satisfy the condition of Theorem 1. Hence,

$$\operatorname{ind}^G A_1 \;-\; \operatorname{ind}^G A_0 \;=\; \operatorname{ind}^G A \;=\; 0.$$

Thus we proved the following

Corollary 1. *The distributional indexes of cobordant transversally elliptic operators coincide.*

Combining this corollary with the discussion in Subsection 1.2 we also obtain the following

COROLLARY 2. $\mathrm{Spin}^{\mathbb{C}}$ *quantization of orbifolds commutes with* $\mathrm{Spin}^{\mathbb{C}}$ *cobordism.*

2.5. The plan of the proof of Theorem 1. We apply the method of [**Brav3**] with necessary modifications.

Choose a G-invariant Riemannian metric on W, which induces the product metric on $U = X \times (-\varepsilon, 0]$. Let $\tilde{W}$ denote the complete non-compact Riemannian manifold obtained from W by attaching the semi-infinite cylinder $X \times [0, \infty)$ to the boundary. We extend the bundle F and the operator B to $\tilde{W}$ in the obvious way.

Consider two linear operators[1] c_L and c_R on the exterior algebra $\Lambda^\bullet\mathbb{C} = \Lambda^0\mathbb{C} \oplus \Lambda^1\mathbb{C}$, defined by the formula

$$c_L\,\omega \;=\; 1 \wedge \omega \;-\; \iota_1\omega; \qquad c_R\,\omega \;=\; 1 \wedge \omega \;+\; \iota_1\omega, \qquad \omega \in \Lambda^\bullet\mathbb{C}. \tag{J.7}$$

(Here we consider 1 as a vector in $\mathbb{C}$ and denote by ι_1 the interior multiplication by this vector.) These operators anti-commute with each other, $c_Lc_R + c_Rc_L = 0$. They also satisfy $c_L^2 = -1$, $c_R^2 = 1$.

Set $\tilde{F} = F \otimes \Lambda^\bullet\mathbb{C}$ and consider the operator

$$\tilde{B} \;:=\; \sqrt{-1}\,B \otimes c_L : C^\infty(\tilde{W}, \tilde{F}) \;\to\; C^\infty(\tilde{W}, \tilde{F}). \tag{J.8}$$

Note, that the operator $\tilde{B}$ is symmetric, since $c_L^* = -c_L$.

Let $p : \tilde{W} \to \mathbb{R}$ be a G-invariant map, whose restriction to $X \times (1, \infty)$ is the projection on the second factor, and such that $p(W) = 0$ (see Subsection 4.1 for a convenient choice of this function). For any $a \in \mathbb{R}$, consider the operator $\mathbf{B}_a := \tilde{B} - (p(x) - a) \otimes c_R$. Then (cf. Lemma 1)

$$\mathbf{B}_a^2 \;=\; B^2 \otimes 1 - R + |p(x) - a|^2, \tag{J.9}$$

where $R : C^\infty(\tilde{W}, \tilde{F}) \to C^\infty(\tilde{W}, \tilde{F})$ is a bounded operator.

Let $\mathbf{B}_a^\pm$ denote the restriction of $\mathbf{B}_a$ to the spaces $F \otimes \Lambda^0\mathbb{C}$ and $F \otimes \Lambda^1\mathbb{C}$ respectively. In Subsection 4.2 we show that, for each $\rho \in \hat{G}$, the index $\mathrm{ind}_\rho^G\,\mathbf{B}_a := \mathrm{Ker}_\rho(\mathbf{B}_a^+) - \mathrm{Ker}_\rho(\mathbf{B}_a^-)$ is well defined and is independent of a.

It follows from (J.9) that, if $a \ll 0$, then the operator $\mathbf{B}_a^2$ is strictly positive. In particular, its kernel is empty and $\mathrm{ind}_\rho^G\,\mathbf{B}_a = 0$. Also, if $a \gg 0$, then all the sections in $\mathrm{Ker}\,\mathbf{B}_a^2$ are concentrated on the cylinder $X \times (0, \infty)$, not far from $X \times \{a\}$ (this part of the proof essentially repeats the arguments of Witten in [**Wi1**]). Hence, the calculation of $\mathrm{Ker}\,\mathbf{B}_a^2$ is reduced to a problem on the cylinder $X \times (0, \infty)$. It is not difficult now to show that $\mathrm{ind}_\rho^G\,\mathbf{B}_a = \mathrm{ind}_\rho^G\,A$ for $a \gg 0, \rho \in \hat{G}$.

Theorem 1 follows now from the fact that $\mathrm{ind}_\rho^G\,\mathbf{B}_a$ is independent of a.

3. Transversally elliptic operators and their indexes

In this section we recall the definitions and some properties of transversally elliptic operators on a compact G-manifold; cf. [**At1**]. Since in the proof of Theorem 1 we apply some of the constructions which Atiyah used to prove that the index of a transversally elliptic operator is well defined, we will briefly recall this proof in Subsection 3.2.

[1] These operators generate two actions of the Clifford algebra of $\mathbb{C}$ on $\Lambda^\bullet\mathbb{C}$, called, respectively, left and right actions. This is the motivation for the subscripts "L" and "R" in our notation.

Throughout the section X is a Riemannian G-manifold, E, F are G-equivariant Hermitian vector bundles over X and $D : C^\infty(X, E) \to C^\infty(X, F)$ is a G-invariant pseudo-differential operator of order 1.

3.1. Transversally elliptic operators. Recall that the leading symbol $\sigma(D)$ of D is a function on the cotangent bundle T^*X taking values in $\mathrm{Hom}(E, F)$. Let $T^*_G X \subset T^*X$ denote the subbundle of covectors which vanish on vectors tangent to the orbits of G. We will identify X with the zero section of $T^*_G X$.

DEFINITION 1. *The operator D is called* transversally elliptic *if $\sigma(D)$ is invertible when restricted to $T^*_G X \backslash X$.*

Fix a bi-invariant Riemannian metric on G, and let $Y_1, \dots, Y_k$ be an orthonormal basis for the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$. Denote by $\tilde{Y}_1, \dots, \tilde{Y}_k$ the corresponding first order differential operators defined by the action of G on E^+, and form the operator

$$\Delta_G \ := \ 1 \ - \ \sum_{i=1}^{k} \tilde{Y}_i^2 : \, C^\infty(X, E) \ \to \ C^\infty(X, E).$$

Consider the second order pseudo-differential operator

$$\bar{D} \ := \ (D, \Delta_G^{1/2}) : \, C^\infty(X, E) \ \to \ C^\infty(X, F) \oplus C^\infty(X, E).$$

One immediately sees that *the operator D is transversally elliptic if and only if the principal symbol of $\bar{D}$ is injective on $T^*X\backslash X$*. Equivalently, the operator

$$\bar{D}^*\bar{D} \ = \ D^*D + \Delta_G : \, C^\infty(X, E) \ \to \ C^\infty(X, E)$$

is elliptic (here $D^*, \bar{D}^*$ denote the formal adjoints of the operators D and $\bar{D}$, respectively).

3.2. The distributional index. Suppose now that $\rho \in \hat{G}$ is an irreducible representation of G and let

$$L^2_\rho(X, E) \ := \ \mathrm{Hom}_G(\rho, L^2(X, E)) \otimes \rho$$

be the ρ-component of the space $L^2(X, E^\pm)$ of square-integrable sections of E. Clearly, the restriction of Δ_G to $L^2_\rho(X, E)$ is bounded by a constant $C(\rho)$. It follows that the space $\mathrm{Ker}_\rho(D) := \mathrm{Ker}\, D \cap L^2_\rho(X, E)$ is a subset of the space

$$(\bar{D}^*\bar{D})_{C(\rho)} \ := \ \big\{ \, u \in L^2_\rho(X, E) : \, \langle \bar{D}^*\bar{D}u, u \rangle \le C(\rho) \, \big\}.$$

By the standard theory of elliptic operators (cf., for example, [**Sh1**] or [**At1**, Lemma 2.3]), the space $(\bar{D}^*\bar{D})_{C(\rho)}$ is finite dimensional. Hence, so is $\mathrm{Ker}_\rho(D)$ and we have the inequality

$$\dim \mathrm{Ker}_\rho(D) \ \le \dim(\bar{D}^*\bar{D})_{C(\rho)}.$$

Similarly, $\dim \mathrm{Ker}_\rho(D^*) \ \le \dim(\bar{D}\bar{D}^*)_{C(\rho)}$. With just a little more work (cf. [**At1**, p. 13]), one shows that the sum (J.3) converges to a distribution on G. Thus the sum (J.4) also converges to a distribution on G, called the *distributional index* of A.

4. Index of the operator $\mathbf{B}_a$

4.1. The calculation of $\mathbf{B}_a^2$. We will use the notation of Subsection 2.5. In particular, $U \simeq X \times (-\varepsilon, 0]$ is a neighborhood of ∂W, $t : U \to (-\varepsilon, 0]$ is the projection, $\tilde{W}$ is the manifold obtained from W by attaching a cylinder, $\tilde{F} = F \otimes \Lambda^\bullet \mathbb{C}$ and $\tilde{B}$ is the operator defined in (J.8). Recall that in Subsection 2.3 we have chosen a connection on F. This connection defines a trivialization of the restriction of F to U along the fibers of t. Hence the Hermitian metric on E induces a metric on $F|_U$. We extend this metric to a G-invariant Hermitian metric on F. This metric induces a Hermitian metric on $\tilde{F}$ in the obvious way.

Let $s : \mathbb{R} \to [0, \infty)$ be a smooth function such that $s(t) = t$ for $|t| \geq 1$, and $s(t) = 0$ for $|t| \leq 1/2$. Consider the map $p : \tilde{W} \to \mathbb{R}$ such that $p(y,t) = s(t)$ for $(y,t) \in X \times (0, \infty)$ and $p(x) = 0$ for $x \in W$. Recall that the operator c_R is defined in (J.7) and define the operator

$$\mathbf{B}_a \;:=\; \tilde{B} \;-\; (p(x) - a) \otimes c_R. \tag{J.10}$$

Lemma 1. *Let $\Pi_i : \tilde{F} \to F \otimes \Lambda^i \mathbb{C}$, $(i = 0, 1)$ be the projections. Then*

$$\mathbf{B}_a^2 \;=\; B^2 \otimes 1 \;-\; R \;+\; |p(x) - a|^2, \tag{J.11}$$

where $R : \tilde{F} \to \tilde{F}$ is a uniformly bounded bundle map, whose restriction to $X \times (1, \infty)$ is equal to $\sqrt{-1}\,\gamma(\Pi_1 - \Pi_0)$, and whose restriction to W vanishes.

Proof. Note, first, that $p(x) - a \equiv -a$ on W. Thus, since c_R anti-commutes with $\tilde{B}$, we have $\mathbf{B}_a^2|_W = \tilde{B}^2|_W + a^2 = B^2 \otimes 1|_W + a^2$. Hence, (J.11) holds, when restricted to W.

We now consider the restriction of $\mathbf{B}_a^2$ to the cylinder $X \times (0, \infty)$. Since the operators c_L and c_R anti-commute, we obtain

$$\mathbf{B}_a^2|_{X \times (0,\infty)} \;=\; B^2 \otimes 1 \;+\; \sqrt{-1}\, s' \gamma \otimes c_L c_R \;+\; |s(t) - a|^2.$$

Since $c_L c_R = \Pi_1 - \Pi_0$, it follows, that (J.11) holds with $R = s' \sqrt{-1}\,\gamma\big(\Pi_1 - \Pi_0\big)$. □

4.2. An estimate on the kernel of $\mathbf{B}_a$. As in Subsection 3.1, we choose an orthonormal basis $Y_1, \dots, Y_k$ for the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$, and we denote by $\tilde{Y}_1, \dots, \tilde{Y}_k$ the corresponding first order differential operators defined by the action of G on $C^\infty(\tilde{W}, \tilde{F})$. Set $\Delta_G := 1 - \sum_{i=1}^k \tilde{Y}_i^2$ and consider the second order pseudo-differential operator

$$\bar{\mathbf{B}}_a \;:=\; (\mathbf{B}_a, \Delta_G^{1/2}) : \, C^\infty(\tilde{W}, \tilde{F}) \;\to\; C^\infty(\tilde{W}, \tilde{F}) \oplus C^\infty(\tilde{W}, \tilde{F}).$$

Using Lemma 1, we have

$$\bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a \;=\; \Big(B^2 \otimes 1 + \Delta_G \Big) \;+\; \Big(|p(x) - a|^2 - R \Big). \tag{J.12}$$

We consider $\bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a$ as an operator acting on the space of square-integrable sections of $\tilde{F}$.

Lemma 2. *The operator $\bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a$ is self-adjoint and has discrete spectrum.*

Proof. Since the operator B is transversally elliptic, the operator (J.12) is elliptic. Hence (cf., for example, [**Sh3**, Lemma 6.3]), the Lemma is equivalent to the following statement: For any $\varepsilon > 0$ there exists a compact set $K \subset \tilde{W}$, such that if u is a smooth compactly supported section of $\tilde{F}$, then

$$\int_{\tilde{W}\setminus K} |u|^2\, d\mu \;<\; \varepsilon \int_{\tilde{W}} \langle \bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a u, u\rangle\, d\mu. \tag{J.13}$$

Here, $d\mu$ is the Riemannian volume element on $\tilde{W}$, and $\langle\cdot,\cdot\rangle$ denotes the Hermitian scalar product on the fibers of $\tilde{F}$.

Set $V(x) = |p(x) - a|^2 - R$. To prove (J.13) note that, since R is bounded, there exists a compact set $K \subset \tilde{W}$, such that $V > 1/\varepsilon$ on $\tilde{W}\setminus K$, i.e.,

$$\int_{\tilde{W}\setminus K} \langle Vu, u\rangle d\mu \;>\; \int_{\tilde{W}\setminus K} |u|^2 d\mu, \quad \text{for all} \quad u \in L^2(\tilde{W}, \tilde{F}).$$

Note, also, that the first summand in (J.12) is a non-negative operator. Hence, we have

$$\int_{\tilde{W}\setminus K} |u|^2\, d\mu \;<\; \varepsilon \int_{\tilde{W}\setminus K} \langle Vu, u\rangle\, d\mu \;\le\; \varepsilon \int_{\tilde{W}} \langle Vu, u\rangle\, d\mu \;\le\; \varepsilon \int_{\tilde{W}} \langle \bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a u, u\rangle\, d\mu.$$

□

We apply the method of Subsection 3.2, to study the kernel of $\mathbf{B}_a$.

For an irreducible representation $\rho \in \hat{G}$, denote by $L^2_\rho(\tilde{W}, \tilde{F})$ the ρ-component of the space of square-integrable sections, and by $\operatorname{Ker}_\rho(\mathbf{B}_a) = \operatorname{Ker} \mathbf{B}_a \cap L^2_\rho(\tilde{W}, \tilde{F})$.

Lemma 3. *The spectrum of the restriction of the operator $\mathbf{B}_a$ to $L^2_\rho(\tilde{W}, \tilde{F})$ is discrete. In particular,* $\dim \operatorname{Ker}_\rho(\mathbf{B}_a) < \infty$.

Proof. As in Subsection 3.2, the equation (J.12) implies that, for each $\rho \in \hat{G}$, there is a constant $C(\rho)$ such that, for all $a, \lambda \in \mathbb{R}$ we have

$$\big\{\, u \in L^2_\rho(X, E) : \langle \mathbf{B}_a^2 u, u\rangle \le \lambda \,\big\} \;\subset\; \big\{\, u \in L^2_\rho(X, E) : \langle \bar{\mathbf{B}}_a^* \bar{\mathbf{B}}_a u, u\rangle \le C(\rho) + \lambda \,\big\}.$$

By Lemma 2, the dimension of the right-hand side of this formula is finite. Hence, so is the dimension of the left-hand side. □

Set $\tilde{F}^+ := F \otimes \Lambda^0\mathbb{C}$, $\tilde{F}^- := F \otimes \Lambda^1\mathbb{C}$, $\mathbf{B}_a^\pm := \mathbf{B}_a|_{L^2(\tilde{W}, \tilde{F}^\pm)}$ and define

$$\operatorname{ind}^G_\rho \mathbf{B}_a \;=\; \dim \operatorname{Ker}_\rho(\mathbf{B}_a^+) \;-\; \dim \operatorname{Ker}_\rho(\mathbf{B}_a^-). \tag{J.14}$$

Remark 2. One can obtain estimates on the growth of the numbers $\operatorname{ind}^G_\rho \mathbf{B}_a$ with ρ and prove that the distributional index $\operatorname{ind}^G \mathbf{B}_a$ is defined. The direct proof of this fact will be more involved than the proof in [**At1**], since we have to work on a non-compact manifold $\tilde{W}$. We will show, however, that $\operatorname{ind}^G_\rho \mathbf{B}_a = 0$ for all $\rho \in \hat{G}, a \in \mathbb{R}$.

Lemma 4. *For each $\rho \in \hat{G}$, the index $\operatorname{ind}^G_\rho \mathbf{B}_a$ is independent of a.*

Proof. From (J.10), we see that $\mathbf{B}_b - \mathbf{B}_a = (b-a)\otimes c_R$ is a bounded operator; depending continuously on $b - a \in \mathbb{R}$. Since $\operatorname{ind}^G_\rho \mathbf{B}_a$ coincides with the usual index of the restriction of $\mathbf{B}^+$ to the space $L^2_\rho(\tilde{W}, \tilde{F})$, the lemma follows from the stability of the index of a Fredholm operator; cf., for example, [**Sh1**, §I.8]. □

Lemma 5. $\operatorname{ind}^G_\rho(\mathbf{B}_a) = 0$ *for all* $\rho \in \hat{G}, a \in \mathbb{R}$.

Proof. By Lemma 4, it is enough to prove the proposition for one particular value of a. But it follows from Lemma 1 that, if a is a negative number such that $a^2 > \sup_{x\in\tilde{W}} \|R(x)\|$, then $\mathbf{B}_a^2 > 0$, so that $\operatorname{Ker}\mathbf{B}_a^2 = 0$. □

To prove Theorem 1 it is enough now to show that $\operatorname{ind}_\rho^G \mathbf{B}_a = \operatorname{ind}_\rho^G A$. This is done in two steps: first, in Section 5, we construct a "model" operator $\mathbf{B}^{\mathrm{mod}}$ on the cylinder $X \times (-\infty, \infty)$, whose index is equal to $\operatorname{ind}_\rho^G A$. Then, in Section 6, we show that $\operatorname{ind}_\rho^G \mathbf{B}_a = \operatorname{ind}_\rho^G \mathbf{B}^{\mathrm{mod}}$.

5. The model operator

The bundles $E^\pm$ lift to Hermitian vector bundles over the cylinder $X \times \mathbb{R}$, which we will denote by the same letters. Consider the Hermitian vector bundle $\tilde{F} := (E^+ \oplus E^-) \otimes \Lambda^\bullet\mathbb{C}$ and the operator $\mathbf{B}^{\mathrm{mod}} : C^\infty(X \times \mathbb{R}, \tilde{F}) \to C^\infty(X \times \mathbb{R}, \tilde{F})$ defined by

$$\mathbf{B}^{\mathrm{mod}} \;:=\; \sqrt{-1}\, A \otimes c_L \;+\; \sqrt{-1}\,\gamma \otimes c_L \frac{\partial}{\partial t} \;+\; t \otimes c_R,$$

where t is the coordinate along the axis of the cylinder. We refer to $\mathbf{B}^{\mathrm{mod}}$ as the *model operator*; cf. [**Sh2**]. It follows from Lemma 3 that the spectrum of the restriction of $\mathbf{B}^{\mathrm{mod}}$ to the space $L^2_\rho(X \times \mathbb{R}, \tilde{F})$ is discrete (To see this, one can set $W = X \times [0, 1]$, and view $X \times \mathbb{R}$ as a manifold obtained from W by attaching a cylinder.)

We define $\operatorname{ind}\mathbf{B}^{\mathrm{mod}}$ by (J.14).

Lemma 6. *The kernel of the model operator* $\mathbf{B}^{mod}$ *is G-equivariantly isomorphic (as a graded space) to* $\operatorname{Ker}(A)$. *In particular,* $\operatorname{ind}_\rho^G \mathbf{B}^{mod} = \operatorname{ind}_\rho^G A$ *for all* $\rho \in \hat{G}$.

Proof. Repeating the arguments of Lemma 2 we see that the model operator $\mathbf{B}^{\mathrm{mod}}$ is self-adjoint. Hence, its kernel coincides with $\operatorname{Ker}{\mathbf{B}^{\mathrm{mod}}}^2$. Also, from Subsection 3.2, we know that the kernel of the transversally elliptic operator A is a direct sum of the kernels of bounded operators $A|_{L^2_\rho(X,E)}$. It follows that $\operatorname{Ker} A = \operatorname{Ker} A^2$. Therefore, to prove the lemma it is enough to show that $\operatorname{Ker}{\mathbf{B}^{\mathrm{mod}}}^2$ is equivariantly isomorphic to $\operatorname{Ker} A^2$.

The same calculations as in the proof of Lemma 1 show that

$$\text{(J.15)}\quad (\mathbf{B}^{\mathrm{mod}})^2|_{L^2(X\times\mathbb{R}, E^\pm\otimes\Lambda^\bullet\mathbb{C})} \;=\; A^2 \otimes 1 \;+\; 1 \otimes \Big(-\frac{\partial^2}{\partial t^2} \pm (\Pi_1 - \Pi_0) + t^2 \Big).$$

Both summands on the right-hand side of (J.15) are non-negative. Hence, the kernel of $(\mathbf{B}^{\mathrm{mod}})^2$ is given by the tensor product of the kernels of these operators.

The space $\operatorname{Ker}\big(-\frac{\partial^2}{\partial t^2} + \Pi_1 - \Pi_0 + t^2\big)$ is one-dimensional and is spanned by the function $\rho^+(t) := e^{-t^2/2} \in \Lambda^0\mathbb{R}$. Similarly, $\operatorname{Ker}\big(-\frac{\partial^2}{\partial t^2} + \Pi_0 - \Pi_1 + t^2\big)$ is one-dimensional and is spanned by the one-form $\rho^-(t) := e^{-t^2/2}ds$, where we denote by ds the generator of $\Lambda^1\mathbb{C}$. It follows that

$$\operatorname{Ker}(\mathbf{B}^{\mathrm{mod}})^2 \cap L^2(X \times \mathbb{R}, E^\pm \otimes \Lambda^\bullet\mathbb{C}) \;\simeq\; \Big\{ \sigma \otimes \rho^\pm(t) :\; \sigma \in \operatorname{Ker} A^2 \cap L^2(X, E^\pm) \Big\}.$$

□

5.1. The shifted model operator. Let $T_a : X \times \mathbb{R} \to X \times \mathbb{R}$, $T_a(x,t) = (x, t+a)$ be the translation, and consider the pull-back map $T_a^* : L^2(X \times \mathbb{R}, \tilde{F}) \to L^2(X \times \mathbb{R}, \tilde{F})$. Set

$$\mathbf{B}_a^{\rm mod} \; := \; T_{-a}^* \circ \mathbf{B}^{\rm mod} \circ T_a^* \; = \; B \otimes 1 \; - \; 1 \otimes c_R\big(t-a\big).$$

Then $\operatorname{ind}_\rho^G \mathbf{B}_a^{\rm mod} = \operatorname{ind}_\rho^G \mathbf{B}^{\rm mod}$, for any $a, \rho \in \mathbb{R}$.

6. Proof of Theorem 1

In this section we fix $\rho \in \hat{G}$. All the operators studied in this section are restricted to the ρ-component of the space of square-integrable sections. For simplicity, we omit the subscript "ρ" from the notion for these operators. In particular, $\mathbf{B}_a^\pm$ denote the restriction of $\mathbf{B}_a$ to the spaces $L^2_\rho(\tilde{W}, \tilde{F}^\pm)$. Similarly, let $\mathbf{B}_\pm^{\rm mod}, \mathbf{B}_{\pm,a}^{\rm mod}$ denote the restriction of the operators $\mathbf{B}^{\rm mod}, \mathbf{B}_a^{\rm mod}$ to the spaces $L^2_\rho(X \times \mathbb{R}, \tilde{F}^\pm)$. Note that, with this notation, we have

$$\begin{aligned} \operatorname{ind}_\rho^G \mathbf{B}_a &= \dim \operatorname{Ker} \mathbf{B}_a^+ - \dim \operatorname{Ker} \mathbf{B}_a^-; \\ \operatorname{ind}_\rho^G \mathbf{B}_a^{\rm mod} &= \dim \operatorname{Ker} \mathbf{B}_{+,a}^{\rm mod} - \dim \operatorname{Ker} \mathbf{B}_{-,b}^{\rm mod}. \end{aligned}$$

If A is a self-adjoint operator with discrete spectrum and $\lambda \in \mathbb{R}$, we denote by $N(\lambda, A)$ the number of the eigenvalues of A not exceeding λ (counting multiplicities).

Proposition 1. *Let $\lambda_\pm$ denote the smallest non-zero eigenvalue of $(\mathbf{B}_\pm^{mod})^2$. Then, for any $0 < \varepsilon < \min\{\lambda_+, \lambda_-\}$, there exists $A = A(\varepsilon, V) > 0$, such that*

$$N\big(\lambda_\pm - \varepsilon, (\mathbf{B}_a^\pm)^2\big) \; = \; \dim \operatorname{Ker}(\mathbf{B}_\pm^{mod})^2, \qquad \text{for all} \quad a > A. \tag{J.16}$$

Before proving the proposition let us explain how it implies Theorem 1.

6.1. Proof of Theorem 1. Let $V_{\varepsilon,a}^\pm \subset L^2_\rho(\tilde{W}, \tilde{F}^\pm)$ denote the vector space spanned by the eigenvectors of the operator $(\mathbf{B}_a^\pm)^2$ with eigenvalues smaller or equal to $\lambda_\pm - \varepsilon$. The operator $\mathbf{B}_a^\pm$ sends $V_{\varepsilon,a}^\pm$ into $V_{\varepsilon,a}^\mp$. It follows that

$$\dim \operatorname{Ker} \mathbf{B}_a^+ \; - \; \dim \operatorname{Ker} \mathbf{B}_a^- \; = \; \dim V_{\varepsilon,a}^+ \; - \; \dim V_{\varepsilon,a}^-.$$

By Proposition 1, the right-hand side of this equality equals $\dim \operatorname{Ker} \mathbf{B}_+^{\rm mod} - \dim \operatorname{Ker} \mathbf{B}_-^{\rm mod}$. Thus $\operatorname{ind}_\rho^G \mathbf{B}_a = \operatorname{ind}_\rho^G \mathbf{B}^{\rm mod}$. Theorem 1 follows now from Lemmas 5 and 6. □

The rest of this section is occupied with the proof of Proposition 1.

6.2. Estimate from above on $N(\lambda_\pm - \varepsilon, (\mathbf{B}_a^\pm)^2)$. We will first show that

$$N(\lambda_\pm - \varepsilon, (\mathbf{B}_a^{\pm)^2})) \le \dim \operatorname{Ker} \mathbf{B}_\pm^{\rm mod}. \tag{J.17}$$

To this end we will estimate the operator $\mathbf{B}_a^2$ from below. We will use the technique of [**Sh2, BF**], adding some necessary modifications.

6.3. The IMS localization. Let $j, \bar{j} : \mathbb{R} \to [0,1]$ be smooth functions such that $j^2 + \bar{j}^2 \equiv 0$ and $j(t) = 1$ for $t \geq 3$, while $j(t) = 0$ for $t \leq 2$. Set $j_a(t) = j(a^{-1/2}t)$, $\bar{j}_a(t) = \bar{j}(a^{-1/2}t)$. These functions induce smooth functions on the cylinder $X \times [0,1]$, which we denote by the same letters. By a slight abuse of notation we will denote by the same letters also the smooth functions on $\tilde{W}$ given by the formulas $j_a(x) = j(a^{-1/2}p(x))$, $\bar{j}_a(x) = \bar{j}(a^{-1/2}p(x))$.

The following version of the IMS localization formula is due to Shubin [**Sh2**, Lemma 3.1] (The abbreviation IMS stands for the initials of R. Ismagilov, J. Morgan, I. Sigal and B. Simon).

Lemma 7. *The following operator identity holds:*

$$\mathbf{B}_a^2 = \bar{j}_a \mathbf{B}_a^2 \bar{j}_a + j_a \mathbf{B}_a^2 j_a + \frac{1}{2}[\bar{j}_a, [\bar{j}_a, \mathbf{B}_a^2]] + \frac{1}{2}[j_a, [j_a, \mathbf{B}_a^2]]. \tag{J.18}$$

Proof. Using the equality $j_a^2 + \bar{j}_a^2 = 1$, we can write

$$\mathbf{B}_a^2 = j_a^2 \mathbf{B}_a^2 + \bar{j}_a^2 \mathbf{B}_a^2 = j_a \mathbf{B}_a^2 j_a + \bar{j}_a \mathbf{B}_a^2 \bar{j}_a + j_a [j_a, \mathbf{B}_a^2] + \bar{j}_a [\bar{j}_a, \mathbf{B}_a^2].$$

Similarly, $\mathbf{B}_a^2 = \mathbf{B}_a^2 j_a^2 + \mathbf{B}_a^2 \bar{j}_a^2 = j_a \mathbf{B}_a^2 j_a + \bar{j}_a \mathbf{B}_a^2 \bar{j}_a - [j_a, \mathbf{B}_a^2] j_a - [\bar{j}_a, \mathbf{B}_a^2] \bar{j}_a$. Summing these identities and dividing by 2, we come to (J.18). □

We will now estimate each of the summands in the right-hand side of (J.18).

Lemma 8. *There exists $A > 0$, such that $\bar{j}_a \mathbf{B}_a^2 \bar{j}_a \geq \frac{a^2}{8} \bar{j}_a^2$, for all $a > A$.*

Proof. Note that $p(x) \leq 3a^{1/2}$ for any x in the support of $\bar{j}_a$. Hence, if $a > 36$, we have $\bar{j}_a^2 |p(x) - a|^2 \geq \frac{a^2}{4} \bar{j}_a^2$.

Set $A = \max\left\{ 36, 4 \sup_{x \in \tilde{W}} |R|^{1/2} \right\}$ and let $a > A$. Using Lemma 1, we obtain

$$\bar{j}_a \mathbf{B}_a^2 \bar{j}_a \ \geq \bar{j}_a^2 |p(x) - a|^2 \ - \ \bar{j}_a R \bar{j}_a \ \geq \ \frac{a^2}{8} \bar{j}_a^2.$$

□

6.4. Let $P_a : L_\rho^2(X \times \mathbb{R}, \tilde{F}) \to \operatorname{Ker} \mathbf{B}_a^{\mathrm{mod}}$ be the orthogonal projection. Let $P_a^\pm$ denote the restriction of P_a to the space $L_\rho^2(X \times \mathbb{R}, \tilde{F}^\pm)$. Then $P_a^\pm$ is a finite rank operator and its rank equals $\dim \operatorname{Ker} \mathbf{B}_{\pm,a}^{\mathrm{mod}}$. Clearly,

$$\mathbf{B}_{\pm,a}^{\mathrm{mod}} + \lambda_\pm P_a^\pm \geq \lambda_\pm. \tag{J.19}$$

By identifying the support of j_a in $X \times \mathbb{R}$ with a subset of $\tilde{W}$, we can and will consider $j_a P_a j_a$ and $j_a \mathbf{B}_a^{\mathrm{mod}} j_a$ as operators on $\tilde{W}$. Then $j_a \mathbf{B}_a^2 j_a = j_a \mathbf{B}_a^{\mathrm{mod}} j_a$. Hence, (J.19) implies the following:

Lemma 9. $j_a \mathbf{B}_a^\pm j_a + \lambda_\pm j_a P_a^\pm j_a \geq \lambda_\pm j_a^2$, $\operatorname{rk} j_a P_a^\pm j_a \leq \dim \operatorname{Ker} \mathbf{B}_\pm^{mod}$.

For an operator $A : L_\rho^2(\tilde{W}, \tilde{F}) \to L_\rho^2(\tilde{W}, \tilde{F})$, we denote by $\|A\|$ its norm.

Lemma 10. *Let* $C = 2 \max \left\{ \max\{|j'(t)|^2, |\bar{j}'(t)|^2\} : t \in \mathbb{R} \right\}$. *Then*

$$\|[j_a, [j_a, \mathbf{B}_a^2]]\| \leq C a^{-1}, \qquad \|[\bar{j}_a, [\bar{j}_a, \mathbf{B}_a^2]]\| \leq C a^{-1}, \qquad \textit{for all} \quad a > 0. \tag{J.20}$$

Proof. From Lemma 1 we obtain

$$|[j_a, [j_a, \mathbf{B}_a^2]]| = 2|j_a'(t)|^2 = 2a^{-1/2} |j'(a^{-1/2}t)|,$$
$$|[\bar{j}_a, [\bar{j}_a, \mathbf{B}_a^2]]| = 2a^{-1/2} |j'(a^{-1/2}t)|.$$

□

From Lemmas 7, 9 and 10 we obtain the following:

COROLLARY 3. *For any $\varepsilon > 0$, there exists $A = A(\varepsilon, V) > 0$, such that, for all $a > A$, we have*

$$\mathbf{B}_a^{\pm} + \lambda_{\pm} j_a P_a^{\pm} j_a \ \geq\ \lambda_{\pm} - \varepsilon, \qquad \mathrm{rk} j_a P_a^{\pm} j_a \leq \dim \mathrm{Ker}\, \mathbf{B}_{\pm}^{mod}. \tag{J.21}$$

The estimate (J.17) follows from Corollary 3 and the following general lemma [**RS**, p. 270]:

LEMMA 11. *Assume that A, B are self-adjoint operators in a Hilbert space $\mathcal{H}$ such that $\mathrm{rk} B \leq k$ and that there exists $\mu > 0$ such that $\langle (A+B)u, u\rangle \geq \mu\langle u, u\rangle$ for any $u \in \mathrm{Dom}(A)$. Then $N(\mu - \varepsilon, A) \leq k$ for any $\varepsilon > 0$.*

6.5. Estimate from below on $N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2)$. To prove Proposition 1 it remains now to show that

$$N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2) \ \geq\ \dim \mathrm{Ker}\, \mathbf{B}_{\pm}^{\mathrm{mod}} \equiv \dim \mathrm{Ker}\, \mathbf{B}_{\pm,a}^{\mathrm{mod}}. \tag{J.22}$$

Let $V_{\varepsilon,a}^{\pm} \subset L_\rho^2(\tilde{W}, \tilde{F})$ denote the vector space spanned by the eigenvectors of the operator $(\mathbf{B}_a^{\pm})^2$ with eigenvalues smaller or equal to $\lambda_{\pm} - \varepsilon$. Let $\Pi_{\varepsilon,a}^{\pm} : L_\rho^2(\tilde{W}, \tilde{F}^{\pm}) \to V_{\varepsilon,a}^{\pm}$ be the orthogonal projection. Then $\mathrm{rk}\Pi_{\varepsilon,a}^{\pm} = N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2)$. As in Subsection 6.4, we can and will consider $j_a \Pi_{\varepsilon,a}^{\pm} j_a$ as an operator on $L_\rho^2(X \times \mathbb{R}, \tilde{F}^{\pm})$. The proof of the following lemma does not differ from the proof of Corollary 3.

LEMMA 12. *For any $\varepsilon > 0$, there exists $A = A(\varepsilon, V) > 0$, such that, for any $a > A$, we have*

$$\mathbf{B}_{\pm,a}^{mod} + \lambda_{\pm} j_a \Pi_a^{\pm} j_a \ \geq\ \lambda_{\pm} - \varepsilon, \qquad \mathrm{rk} j_a \Pi_a^{\pm} j_a \leq \dim N(\lambda_{\pm} - \varepsilon, (\mathbf{B}_a^{\pm})^2). \tag{J.23}$$

The estimate (J.22) follows now from Lemmas 12 and 11.

The proof of Proposition 1 is complete. □

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