APPENDIX C

Equivariant cohomology

1. The definition and basic properties of equivariant cohomology

1.1. The topological definition of equivariant cohomology. Let G be a compact Lie group. There exists a numerable[1] principal G-bundle $EG \to BG$ whose total space EG is contractible. The *equivariant cohomology* of a G-manifold M is defined to be the (say, singular) cohomology of the *Borel construction* $EG \times_G M$:

$$H^*_G(M) = H^*(EG \times_G M). \tag{C.1}$$

It is well defined because $EG \times_G M$ is unique up to homotopy equivalence.

If G acts freely, $H^q_G(M) = H^q(M/G)$ for all q, since $EG \times_G M$ is a fiber bundle over M/G with a contractible fiber EG.

If G does not act freely, this is no longer true. For instance, the equivariant cohomology of a point is equal to the cohomology of the classifying space, $BG = EG/G$, which is usually infinite–dimensional.

For more details, see [**Hus**] and [**Bor2, Dol1**].

Example C.1. For $G = S^1$, one can take $EG = S^\infty$, interpreted as the direct limit of odd–dimensional spheres $EG_k = S^{2k+1} \subset \mathbb{C}^{k+1}$ with respect to the natural inclusions, and $BG = \mathbb{CP}^\infty = \varinjlim \mathbb{CP}^k$. For $G = \mathrm{U}(n)$, we obtain EG as the direct limit of the *Stiefel manifolds* of unitary n frames in $\mathbb{C}^k$. For a general G, one can take the Stiefel manifold with the G-action induced by a faithful representation $G \to \mathrm{U}(n)$. In all these cases, $EG \times_G M$ is a direct limit of finite–dimensional manifolds, $EG_k \times_G M$, with respect to natural inclusions. For every degree q, we have

$$H^q_G(M) = H^q(EG_k \times_G M) \tag{C.2}$$

for all sufficiently large k. This is established by a routine argument from the following two facts:

- BG_{k+1} is obtained from BG_k by attaching cells of dimension $\geq q$ if k is sufficiently large.
- for each k there exists an open neighborhood of EG_k in EG which equivariantly strongly deformation retracts to EG_k.

The second assertion can be seen by embedding EG in the Stiefel manifold of n-frames in a Hilbert space.

In particular, for $G = S^1$,

$$H^*_G(\text{point}) = H^*(BG)$$

[1] *Numerable* means that there exists a partition of unity subordinate to an open covering $\{U_i\}$ such that the bundle is trivial over each U_i.

is the space of polynomials in one variable, $\mathbb{R}[u]$, with degree $u = 2$. When $G \cong (S^1)^r$ is a torus, we can take $EG = (S^\infty)^r$ and $H^*(BG) = \mathbb{R}[u_1, \dots, u_r]$; see [**BT1**, §18].

1.2. The Cartan model. Let G be a compact connected Lie group acting smoothly on a compact smooth manifold M. Then, over $\mathbb{R}$, the equivariant cohomology of M is equal to the cohomology of a differential complex, an equivariant version of the de Rham complex called the *Cartan model*, which we will now describe.

The *equivariant differential forms* of degree q are, by definition, the elements of the space

$$\Omega^q_G(M) = \bigoplus_{2i+j=q} (S^i(\mathfrak{g}^*) \otimes \Omega^j(M))^G, \tag{C.3}$$

i.e., G-equivariant polynomial functions $\alpha\colon \mathfrak{g} \to \Omega^*(M)$ on the Lie algebra $\mathfrak{g}$ taking values in the space of differential forms on the manifold. Notice the grading: if $\alpha = P \otimes \omega$, where P is a real valued homogeneous polynomial on $\mathfrak{g}$ and ω is a real valued differential form on M, then $\deg(\alpha) = 2\deg(P) + \deg(\omega)$.

Equivariance means that for all $\xi \in \mathfrak{g}$ and $b \in G$,

$$\alpha(\mathrm{Ad}(b)\xi) = b^*\alpha(\xi).$$

In particular, setting $b = \exp(t\xi)$ and taking the derivative at $t = 0$, we see that

$$0 = L_{\xi_M}\alpha(\xi). \tag{C.4}$$

Hence, the differential form $\alpha(\xi)$ is invariant under the action of the one-parameter group generated by ξ. In general, however, the forms $\alpha(\xi)$ are not invariant under the action of the whole group G if G is not abelian.

EXAMPLE C.2. Let us identify $\mathrm{Lie}(S^1)$ with $\mathbb{R}$ when $G = S^1$. Then an equivariant differential form is a finite sum $\sum \alpha_j u^j$ where u is a formal variable and α_j are invariant differential forms.

The equivariant exterior derivative,

$$d_G\colon \Omega^q_G(M) \to \Omega^{q+1}_G(M),$$

is defined by

$$(d_G\alpha)(\xi) = d(\alpha(\xi)) + \iota(\xi_M)\alpha(\xi).$$

LEMMA C.3. $d_G^2 = 0$

PROOF. $(d_G^2\alpha)(\xi) = d\iota(\xi_M)\alpha(\xi) + \iota(\xi_M)d\alpha(\xi) = L_{\xi_M}\alpha(\xi) = 0$ by (C.4). □

EXERCISE. Prove that $\ker d_G/\mathrm{im}\, d_G = S^*(\mathfrak{g}^*)^G$ when $M = \{\text{point}\}$.

The cohomology of the differential complex $(\Omega^*_G(M), d_G)$ turns out to be the same as the equivariant cohomology of M with real coefficients:

THEOREM C.4 (Equivariant de Rham Theorem). *Let G be a compact connected Lie group and let M be a G-manifold. Then*

$$H^*_G(M; \mathbb{R}) = \ker d_G/\mathrm{im}\, d_G. \tag{C.5}$$

In particular, the cohomology of the classifying space BG is the ring of Ad-invariant polynomials on the Lie algebra

$$\mathcal{R}_G := H^*_G(\text{point}) = H^*(BG) = S^*(\mathfrak{g}^*)^G. \tag{C.6}$$

In reality this fact is usually established as the first step of the proof of the equivariant de Rham theorem, not as a consequence of the theorem.

The equivariant de Rham theorem, as well as the notion of the Cartan model, are due to H. Cartan [**Car1, Car2**]. We will outline a proof of the equivariant de Rham theorem in Section 3.

The Cartan model readily generalizes to a pure algebraic setting, giving the cohomology of the so-called G-differential complexes. The reader interested in this construction and in more details on the Cartan model may consult, for example, [**Car1, Car2, BV1, DKV, Gin5, GS8**] and [**GLS**, Appendix B]. For a topological treatment, see [**Hs, AB2**] and references therein.

1.3. Basic properties of equivariant cohomology and examples.

PROPOSITION C.5 (Module structure).

1. *A G-equivariant map $f\colon M \to N$ induces pullback maps on differential forms and cohomology, $f^*\colon \Omega^*_G(N) \to \Omega^*_G(M)$ and $f^*\colon H^*_G(N) \to H^*_G(M)$.*
2. *In particular, the map $M \to \{point\}$ gives rise to an $\mathcal{R}_G$-module structure on $\Omega^*_G(M)$ and on $H^*_G(M)$, so that $\oplus_q \Omega^q_G(M)$ and $\oplus_q H^q_G(M)$ are graded $\mathcal{R}_G$-modules.*
3. *For any G-equivariant homotopy, $f_t\colon M \to N$, $t \in [0,1]$, the pullback maps on cohomology, $f_t\colon H^*_G(N) \to H^*_G(M)$, are the same for all t.*

EXERCISE. Prove these properties in the Cartan (differential) model (C.5) (with f and f_t smooth) and in the Borel (topological) model (C.1).

EXAMPLE C.6 (Trivial action). Assume that the G-action on M is trivial. Then $H^*_G(M)$ is canonically isomorphic to $H^*(M) \otimes \mathcal{R}_G$ as an algebra over $\mathcal{R}_G$. This follows directly from the definitions, either in the Cartan model or in the topological setting. Equivalently, $H^*_G(M)$ is the space of G-invariant polynomials on $\mathfrak{g}$ with values in $H^*(M)$.

EXAMPLE C.7 (Zeroth equivariant cohomology). The space $\Omega^0_G(M)$ is comprised of smooth invariant functions $f\colon M \to \mathbb{R}$. Since $d_G f = df$, the closed 0-forms are locally constant invariant functions. Hence, $H^0_G(M) = \mathbb{R}^k$, where k is the number of connected components of M/G.

EXAMPLE C.8 (First equivariant cohomology). The space $\Omega^1_G(M)$ consists of invariant one-forms α on M. Since $(d_G\alpha)(\xi) = d\alpha + \iota(\xi_M)\alpha$, we have $d_G\alpha = 0$ if and only if α is closed and horizontal. The exact equivariant one-forms are df, where f is invariant. Hence,

$$H^1_G(M) = \frac{\text{closed basic forms}}{\text{exact basic forms}} = H^1(M/G),$$

by Corollary B.36.

EXAMPLE C.9 (Equivariant two-forms). Equivariant two-forms are sums $\omega+\Phi$, where ω is an invariant two-form and $\Phi \in \mathfrak{g}^* \otimes \Omega^0(M)$ is invariant. Equivalently, Φ is an equivariant smooth function from M to $\mathfrak{g}^*$. Denote $\Phi^\xi = \langle \Phi, \xi \rangle$. The exterior derivative, $d_G(\omega+\Phi)(\xi) = d\omega + \iota(\xi_M)\omega + d\Phi^\xi$, is zero if and only if ω is closed and for all $\xi \in \mathfrak{g}$

$$\iota(\xi_M)\omega = -d\Phi^\xi. \tag{C.7}$$

Example C.10 (Second equivariant cohomology). In this example we use the notion of Hamiltonian assignments introduced in Appendix E. By Proposition E.38, for a torus action we have a short exact sequence

$$0 \to H^2(M/G) \to H^2_G(M) \to \left\{ \begin{array}{c} \text{Hamiltonian} \\ \text{assignments} \end{array} \right\} \to 0.$$

Here the Hamiltonian assignment of the class $[\omega+\Phi]$ associates the element $A(X) = \Phi^H(X)$ of $\mathfrak{h}^*$ to an orbit type stratum X of M with stabilizer $H = G_X$.

Example C.11 (Vector spaces). If G acts linearly on a vector space V, two equivariant differential forms are in the same cohomology class if and only if their restrictions to the origin are equal.

This follows immediately from the fact that the origin is an equivariant deformation retract of the vector space.

In what follows we will need a more explicit description of the coefficient ring $\mathcal{R}_G = H^*(BG) = (S^*\mathfrak{g}^*)^G$.

Proposition C.12 (Chevalley's Theorem). *The restriction to a maximal commutative subalgebra $\mathfrak{t} \subset \mathfrak{g}$ gives rise to an isomorphism of algebras*

$$(S^*\mathfrak{g}^*)^G \to (S^*\mathfrak{t}^*)^W,$$

where W is the Weyl group.

Since $\mathfrak{t}$ meets every orbit of the adjoint action, the only non-obvious part of this proposition is that the restriction map is onto. This is a well-known (but not entirely trivial) result from the theory of Lie groups and their representations. We refer the reader to [**Di**, Theorem 7.3.5] and [**Va**, Theorem 4.9.2] for the proof. Here, instead, we give a simple proof of this fact for $G = \mathrm{U}(n)$.

Example C.13. Let $G = \mathrm{U}(n)$ and let us, for the sake of simplicity, consider complex valued polynomials. By definition, $(S^*\mathfrak{g}^*)^G$ is the algebra of polynomials on $\mathfrak{u}(n)$ invariant under conjugation. Let us take as $\mathfrak{t}$ the space of diagonal matrices in $\mathfrak{u}(n)$. We will regard the diagonal entries as coordinates $(x_1, \dots, x_n)$ on $\mathfrak{t}$. Then W is S_n, the group of permutations, and $(S^*\mathfrak{t}^*)^W$ is just the algebra of symmetric polynomials in $x_1, \dots, x_n$. As is well known, such polynomials can be expressed as polynomials f of elementary symmetric functions $\sigma_1, \dots, \sigma_n$. It is clear that f can be extended to a continuous G-invariant function on $\mathfrak{u}(n)$ by simply replacing the symmetric functions in $x_1, \dots, x_n$ by the symmetric functions in the eigenvalues. Denote these functions by σ_k again. To show that this extension is a polynomial, it suffices to prove that all σ_k are polynomials, but this follows immediately from the fact that $\sigma_k(A) = \operatorname{trace} \wedge^k A$ for any matrix A.

Remark C.14. As we have pointed out above, elements of $\Omega^*_G(M)$ can be thought of as G-invariant polynomial functions on $\mathfrak{g}$ with values in $\Omega^*(M)$. By replacing the ring of polynomials on $\mathfrak{g}$ by some other class of functions, we may still obtain complex, and hence cohomology spaces. These spaces are often used as repositories for equivariant cohomology classes that are not polynomial on $\mathfrak{g}$. This is the case, for instance, with the equivariant Todd class Td^G; see Section 2 of Appendix I. The classes of "functions" usually considered are

- formal power series on $\mathfrak{g}$;
- the germs at 0 of analytic functions on $\mathfrak{g}$;
- smooth functions on $\mathfrak{g}$, or their germs at 0;
- distributions on $\mathfrak{g}$.

For formal power series, the resulting cohomology can also be described in pure topological terms, using the inverse limit construction in the setting of Example C.1. For other classes of functions, one works directly with the Cartan model, with polynomials replaced by a suitable class A of "functions". We will refer to the resulting cohomology as the *equivariant cohomology over* A. Under suitable hypotheses (see, e.g., [**DKV**, Corollaries 64 and 104]), the equivariant cohomology over A is the extension of the standard equivariant cohomology, i.e., is equal to $H^*_G(M) \otimes_{\mathcal{R}_G} A$. A majority of results presented below for $H^*_G(M)$ extend with appropriate modifications to cohomology over other classes of "functions". The reader interested in the precise definitions and results should consult [**DKV**].

2. Reduction and cohomology

The projection $EG\times_G M \to M/G$ induces the pull-back algebra homomorphism

$$H^*(M/G) \to H^*_G(M). \tag{C.8}$$

If the G-action is free, this map is an isomorphism. The same is true for cohomology with *real* coefficients if G acts *locally* freely, i.e., with finite stabilizers. This can be established in the topological model using, for example, a spectral sequence argument, or it can be proved directly in the Cartan model. Note that, in general, the map (C.8) need not be onto (e.g., take M to be a point), nor one-to-one (see Example C.18 below).

The spectral sequence argument goes as follows. Consider the projection map $\rho\colon EG \times_G M \to M/G$. Its "fiber" over $x \in M/G$ is EG/G_x, the classifying space BG_x of the stabilizer G_x of x. The Leray spectral sequence of ρ (see [**Bor2**] or [**Hs**, Section III.1]) with real coefficients collapses in the E_2-term because G_x is finite, and hence $H^{*>0}(BG_x;\mathbb{R}) = 0$. Therefore, $H^*(EG \times_G M;\mathbb{R}) = H^*(M/G;\mathbb{R})$. (See, e.g., [**Gin3**, Lemma 5.3] for more details.) The direct proof, due to Cartan, is somewhat involved. (See [**Car1, Car2**], also [**DKV**] (Theorem 17 of Part I with the ring of polynomials on $\mathfrak{g}^*$ replaced by smooth functions), [**GLS**, Appendix B], and [**GS8**].) The special case of $H^2_{S^1}$ is particularly instructive:

Lemma C.15. *Let $G = S^1$ act on M locally freely. Let $\pi\colon M \to M/S^1$ be the quotient map. Define $\pi^*\colon H^2(M/S^1) \to H^2_{S_1}(M)$ by $[\alpha] \mapsto [\pi^*\alpha]$. Then π^* is well defined, one-to-one, and onto.*

Remark C.16. The quotient M/S^1 is an *orbifold*. Locally, an orbifold is the quotient of $\mathbb{R}^n$ by a finite group, and a differential form is given by an invariant differential form on $\mathbb{R}^n$. Just like for manifolds, the cohomology (over $\mathbb{R}$) of an orbifold is $\ker d/\mathrm{image}\, d$ on differential forms. In the statement of the lemma, $[\alpha]$ means the cohomology class represented by the differential form α on M/S^1.

A differential form β on M descends to a differential form on the orbifold M/G if and only if β is *basic*, i.e., invariant and horizontal ($\iota(\xi_M)\beta = 0$ for all $\xi \in \mathfrak{g}$). See Corollary B.36.

Proof of Lemma C.15. The pullback $\pi^*\alpha$ is a closed and basic differential form on M, hence it is an equivariantly closed equivariant differential form (which happened to be independent of the formal variable u).

The pullback homomorphism π^* is well defined on cohomology, because the pullback of any d-exact form is d_{S^1}-exact: $\pi^* d\beta = d_{S^1}\pi^*\beta$.

Let us show that π^* is one-to-one in cohomology. Suppose that $\pi^*\omega$ is d_{S^1}-exact. This means that

$$\pi^*\omega = d_{S^1}\beta = d\beta + u\iota(\xi_M)\beta$$

for some invariant one-form β. Since there is no variable u on the left-hand side, β is horizontal: $\iota(\xi_M)\beta = 0$. Since β is horizontal and invariant, it is basic. Let b be the one-form on M/S^1 whose pullback to M is β. Then $\pi^*\omega = d(\pi^* b)$, and $\omega = db$ is exact.

Let us show that π^* is onto. Let $\omega + u\Phi$ be an arbitrary equivariantly closed equivariant two-form on M. Then ω is an invariant two-form, Φ is an invariant function, and

$$d_{S^1}(\omega + u\Phi) = d\omega + u(d\Phi + \iota(\xi_M)\omega) = 0.$$

If $\Phi \equiv 0$, this implies that ω is closed and basic. Hence, ω is the pullback of a closed two-form on M/S^1. If $\Phi \not\equiv 0$, we subtract an equivariantly exact two-form to obtain a new form with $\Phi_{\text{new}} \equiv 0$. Namely, let γ be a connection one-form on M, i.e., a one-form satisfying

(1) $\iota(\xi_M)\gamma \equiv 1$, and
(2) $L_{\xi_M}\gamma \equiv 0$.

Then

$$d_{S^1}(\Phi\gamma) = d(\Phi\gamma) + u\iota(\xi_M)(\Phi\gamma) = \omega' + u\Phi$$

for some ω'. To complete the proof it suffices to subtract this from $\omega + u\Phi$. □

Example C.17. When the G-action is locally free, but not free, the map $\pi^*\colon H^*(M/G;\mathbb{Z}) \to H^*_G(M;\mathbb{Z})$ may fail to be an isomorphism. For example, let S^1 act on the unit sphere $S^3 \subseteq \mathbb{C}^2$ by $a\cdot(z,w) = (az, a^2w)$. Then $H^2(S^3/S^1;\mathbb{Z})$ pulls back to the subgroup of index two in $H^2_{S^1}(S^3;\mathbb{Z}) \cong \mathbb{Z}$, as one can check by a Mayer-Vietoris argument. Alternatively, consider a non-effective action of $G = S^1$ on M which factors into the double cover $G \to S^1$ followed by a free action of S^1 on M. Then $H^*(M/G;\mathbb{Z})$ is a subgroup of index two in $H^2_G(M;\mathbb{Z})$. This follows from the Leray spectral sequence for the fiber bundle $EG \times_G M \to M/G$ whose fiber is $B\mathbb{Z}_2 = \mathbb{RP}^\infty$.

Example C.18. As we have pointed out above, in general, the pull-back map $H^*(M/G) \to H^*_G(M)$ need not be one-to-one. The first degree in which this phenomenon can occur is three. Let us illustrate this by an example.

Let M be the unit sphere S^4 in $\mathbb{R}\times\mathbb{C}^2$ and $G = S^1$ with its standard diagonal action on $\mathbb{C}^2$ and trivial action on $\mathbb{R}$. Using the Mayer–Vietoris exact sequence, it is easy to see that $H^3_G(M) = 0$. (In fact, M is equivariantly formal and hence $H^*_G(M) = H^*(M)\otimes\mathcal{R}_G$; see Section 4 below. This, for example, can be shown using the results of Appendix G. Indeed, the height function on S^4 is a non-degenerate abstract moment map for the action. Now formality follows from Theorem G.9. On the other hand, M/G is homeomorphic to S^3 and thus $H^3(M/G) = \mathbb{R}$. Therefore, the map

$$\mathbb{R} = H^3(M/G) \to H^3_G(M) = 0$$

is not a monomorphism.

3. Additivity and localization

3.1. Additivity techniques. An important feature of (co)homology is that information about the cohomology of a space can be recovered from the cohomology of its smaller pieces. This property of cohomology allows one, under favorable conditions, to determine the cohomology of the space, or at least to obtain some valuable information about it. One way to do this is by using the long exact sequence of a pair. Another way to state the same principle is to say that cohomology is *additive*, where additivity is understood in a very broad sense. More formally, this is expressed by the Mayer–Vietoris exact sequence. These two exact sequences, which are essentially equivalent to each other (see [**Sw**]), are among the most frequently used and conceptually significant properties of (co)homology.

The long exact sequence of a pair and the Mayer–Vietoris exact sequence also exist for equivariant cohomology and play a role similar to that of their ordinary versions.

Namely, as follows immediately from the topological description of equivariant cohomology, there is a long exact sequence associated with a pair of G-spaces. To be more precise, let Z be a G-invariant subset of a G-space M. (For example, M is a G-manifold with boundary and $Z = \partial M$.) Set $H^*_G(M, Z) = H^*(EG \times_G M, EG \times_G Z)$. Then the long exact sequence for the ordinary cohomology of the pair $(EG \times_G M, EG \times_G Z)$ turns into the exact sequence

$$\ldots \to H^*_G(M, Z) \to H^*_G(M) \to H^*_G(Z) \to \ldots .$$

In the same vein, let U and V be G-invariant subsets of M such that $M = \operatorname{int} U \cup \operatorname{int} V$. Then we have the exact sequence

$$\ldots \to H^*_G(M) \to H^*_G(U) \oplus H^*_G(V) \to H^*_G(U \cap V) \to \ldots ,$$

called the *Mayer–Vietoris exact sequence* for equivariant cohomology.

Remark C.19. For the Mayer–Vietoris exact sequence to hold, it is enough to assume that U, V, and $U \cap V$ are, respectively, equivariant deformation retracts of U', V', and $U' \cap V'$, where U' and V' are open and $M = U' \cup V'$.

As does the long exact sequence for pairs of spaces, the Mayer–Vietoris exact sequence for equivariant cohomology follows directly from its ordinary, non-equivariant, version. (See, e.g., [**Mas3**, p. 58] or [**Sw**].) Alternatively, when U and V are open it can be proved from the Cartan model in the same way as the ordinary Mayer–Vietoris exact sequence is proved for de Rham cohomology. (See, e.g., [**BT1**].) Finally, the Mayer–Vietoris exact sequence extends to pairs of subsets (triads) in a natural way [**Sw**].

3.2. An application: the equivariant de Rham theorem. As an application of the additivity techniques, we outline the proof of Theorem C.4, which asserts that the cohomology of the Borel construction $EG \times_G M$, over $\mathbb{R}$, is isomorphic to the cohomology of the Cartan complex, $\ker d_G/\operatorname{im} d_G$.

Assume first that M is a point. Then we need to prove (C.6), which is done in Section 5.2.

More generally, let M be an orbit, i.e., a homogeneous space G/K, where K is a closed subgroup of G. The space $EG \times_G M$ is then homotopy equivalent to BK, and hence $H^*_G(M) = H^*(BK)$. On the other hand, a somewhat non-trivial algebraic calculation (see, e.g., [**DKV**, pp. 53–57] or [**GLS**, Appendix B]) shows that the cohomology in the Cartan model is $(S^*\mathfrak{k}^*)^K$. (For example, if $M = G$, the complex $\Omega^*_G(G)$ is simply the Weil complex $\Lambda^*\mathfrak{g}^* \otimes S^*\mathfrak{g}^*$ which is known to be acyclic.) Now, using (C.6) with K in place of G, we obtain the equivariant de Rham theorem for M. (The reader interested in more recent results about the equivariant cohomology of homogeneous spaces should consult [**BT3**].)

Passing to the general case, we adapt the proof of the de Rham theorem for ordinary cohomology given in [**BT1**]. Assume, for the sake of simplicity, that M is compact. Then we can cover M by a finite number of invariant open sets such that each non-empty multiple intersection is (smoothly) equivariantly retractable to an orbit. The equivariant de Rham theorem holds for all intersections of open sets from the cover since the equivariant cohomology is an invariant of smooth equivariant homotopy, both in the topological model and in the Cartan model (see Proposition C.5). Now the proof is finished in the same way as for the ordinary de Rham theorem in [**BT1**], but using the five–lemma and the Mayer–Vietoris sequences for the equivariant cohomology in the topological model and in the Cartan model.

3.3. Borel's Localization. A key feature of equivariant cohomology which has no analog for ordinary cohomology is that a substantial part of the structure of $H^*_G(M)$ can be recovered, when G is abelian, from the fixed point set. One of the incarnations of this general principle is the following, slightly weakened, version of a theorem due to Borel, [**Bor2**].

THEOREM C.20 (Borel's localization theorem). *Let M be a compact manifold, possibly with boundary, acted upon by a torus G. Then the restriction homomorphism*

$$H^*_G(M) \to H^*_G(M^G)$$

is an isomorphism modulo torsion over the ring $\mathcal{R}_G$ of Ad-invariant polynomials on $\mathfrak{g}$. Moreover, if the G-action on ∂M has no fixed points, the restriction homomorphism

$$H^*_G(M, \partial M) \to H^*_G(M^G)$$

is also an isomorphism modulo $\mathcal{R}_G$-torsion.

Since $H^*_G(M^G) = H^*(M^G) \otimes \mathcal{R}_G$, this, in particular, implies that

$$\mathrm{rk}_{\mathcal{R}_G} H^*_G(M) = \dim H^*(M^G). \tag{C.9}$$

Note that a torsion $\mathcal{R}_G$-module may be quite large since $\mathcal{R}_G$ is the ring of polynomials. For example, if $G = S^1$ acts freely on M, the entire space $H^*_{S^1}(M) = H^*(M/S^1)$ is a torsion $\mathcal{R}_G$-module.

Theorem C.20 is a special case of the following result, in which we take either $Z = \partial M$ or $Z = \emptyset$.

THEOREM C.21. *Let a torus G act on a compact manifold M, possibly with boundary, and let $Z \subset M$ be a subset which is itself a compact manifold. Then the kernel and cokernel of the restriction homomorphism*

$$H^*_G(M, Z) \to H^*_G(M^G, Z^G)$$

are $\mathcal{R}_G$-torsion modules. In other words, the restriction homomorphism is an isomorphism modulo $\mathcal{R}_G$-torsion.

Let us outline a simple proof of Theorem C.21 using the Mayer–Vietoris exact sequence as in [**AB2**]. This method also provides some information on the torsion of $H_G^*(M)$. The reader interested in more details should consult [**AB2**].

PROOF. First observe that $H_G^*(M)$ is an $\mathcal{R}_G$-torsion module when the G-action has no fixed points on M. To see this, let us cover M by a finite number of G-invariant open sets of the form $G \times_K B$, where K is a subgroup of G, and the slice B is diffeomorphic to a subset of a vector space with a linear K-action. The existence of such a cover follows from the slice theorem. (See Appendix B and, in particular, Theorem B.24.)

Each $H_G^*(G \times_K B)$ is a torsion $\mathcal{R}_G$-module. Indeed, $H_G^*(G \times_K B) = H^*(EG \times_G G \times_K B) = H^*(EG \times_K B) = H_K^*(B)$, where for the last equality we take EK to be EG with the action restricted to K. Since the action of $\mathcal{R}_G$ on this space factors through the restriction $\mathcal{R}_G \to \mathcal{R}_K$, every Ad-invariant polynomial on $\mathfrak{g}$ that vanishes on $\mathfrak{k}$ annihilates $H_G^*(G \times_K B)$.

Any G-invariant open subset of $G \times_K B$ is again of the form $G \times_K B'$, where the slice B' is obtained as the intersection of the slice B with the open subset. Therefore, the equivariant cohomology of the subset is again a torsion $\mathcal{R}_G$-module.

By the Mayer–Vietoris exact sequence, $H_G^*(M)$ is a torsion $\mathcal{R}_G$-module. Indeed, let V be a union of a finite number of sets of the form $G \times_K B$, and let V' be an additional such set. The Mayer–Vietoris exact sequence for V and V' implies that

$$H_G^{*-1}(V \cap V') \to H_G^*(V \cup V') \to H_G^*(V) \oplus H_G^*(V')$$

is exact. The above reasoning shows that V' and $V \cap V'$ are torsion $\mathcal{R}_G$-modules. Arguing inductively, let us assume that $H_G^*(V)$ is a torsion $\mathcal{R}_G$-module. By exactness, the middle term, $H_G^*(V \cup V')$, is also a torsion $\mathcal{R}_G$-module.

Let us proceed by induction in the number of components of the fixed point set of M. We have just proved the theorem if this number is zero. Pick a connected component F of the fixed point set M^G, let U be a small neighborhood of F, equivariantly contractible to F, and let $V = M \smallsetminus U$. The Mayer–Vietoris exact sequences for $M = V \cup \overline{U}$ and $M^G = V^G \sqcup F$ read

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H_G^*(M) & \longrightarrow & H_G^*(V) \oplus H_G^*(\overline{U}) & \longrightarrow & H_G^*(\partial \overline{U}) & \longrightarrow & \dots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\dots & \longrightarrow & H_G^*(M^G) & \longrightarrow & H_G^*(V^G) \oplus H_G^*(F) & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Suppose that the theorem is true for manifolds whose fixed point set has fewer connected components than M^G does. Then all vertical arrows but the first one are isomorphisms modulo $\mathcal{R}_G$-torsion (i.e., the kernels and cokernels of these homomorphisms are torsion modules). The five-lemma implies that the first arrow is also an isomorphism modulo torsion.

For the relative equivariant cohomology, $H_G^*(M, Z)$, the result follows immediately from the long exact sequence for the pair (M, Z). $\square$

4. Formality

In this section we introduce the notion of formality and prove some properties of formal G-manifolds, to be used later in Appendix G.

4.1. Definitions of formality. Throughout this section we consider only cohomology with real coefficients. As above, we denote $H^*(BG)$ by $\mathcal{R}_G$.

DEFINITION C.22. The pair (M, Z) is *formal* if $H_G^*(M, Z)$ is isomorphic to $H^*(M, Z) \otimes \mathcal{R}_G$ as an $\mathcal{R}_G$-module:

$$H_G^*(M, Z) \simeq H^*(M, Z) \otimes \mathcal{R}_G \quad \text{over} \quad \mathcal{R}_G. \tag{C.10}$$

Thus, the $\mathcal{R}_G$-module structure on $H_G^*(M, Z)$ does not see the G-action; the action appears "trivial" to it. In particular, $H_G^*(M, Z)$ has no $\mathcal{R}_G$-torsion.

LEMMA C.23. *Let M be a G-manifold, let U and V be invariant open subsets of M such that $M = U \cup V$, or such that U and V satisfy the hypothesis of Remark C.19. Let $Z = U \cap V$. Suppose that the pairs (U, Z) and (V, Z) are formal. Then so is the pair (M, Z).*

PROOF. Consider the following segment of the Mayer–Vietoris exact sequence

$$H_G^{*-1}(Z, Z) \to H_G^*(M, Z) \to H_G^*(U, Z) \oplus H_G^*(V, Z) \to H_G^*(Z, Z).$$

Since $H_G^*(Z, Z) = 0$,

$$H_G^*(M, Z) = H_G^*(U, Z) \oplus H_G^*(V, Z).$$

If the right-hand side is formal, so is the left. ☐

The Serre spectral sequence of the fibration $EG \times_G M \to BG$ converges to its E_∞-term, which is the graded $\mathcal{R}_G$ algebra associated with a certain filtration of the cohomology $H_G^*(M, Z)$. As an $\mathcal{R}_G$-module, E_∞ is *equal* to $H_G^*(M, Z)$. The E_2-term of this spectral sequence is $H^*(M) \otimes \mathcal{R}_G$. Similarly, for a pair (M, Z) of G-spaces, there is a spectral sequence converging to $H_G^*(M, Z)$ whose E_2-term is $H^*(M, Z) \otimes \mathcal{R}_G$.

Recall that the E_{n+1}-term of a spectral sequence is the cohomology of the E_n-term with respect to some differential d_n. By definition, a spectral sequence collapses in the E_2-term if the differentials d_n vanish for all $n \geq 2$.

LEMMA C.24. *The pair (M, Z) is formal if and only if the spectral sequence collapses at the E_2-term or, equivalently, if* $\operatorname{rk}_{\mathcal{R}_G} H_G^*(M, Z) = \dim H^*(M, Z)$.

PROOF. If the spectral sequence collapses at the E_2-term, we obviously have $E_\infty = E_2$. Because $E_\infty = H_G^*(M, Z)$ and $E_2 = H^*(M, Z) \otimes \mathcal{R}_G$ as $\mathcal{R}_G$ modules, this implies formality. Formality clearly implies that $\operatorname{rk}_{\mathcal{R}_G} H_G^*(M, Z) = \dim H^*(M, Z)$.

Let us now assume that the spectral sequence does not collapse at the E_2 term. Let n be such that the differentials $d_2, \ldots, d_{n-1}$ are all zero and $d_n \neq 0$. Then $E_n = \ldots = E_2 = H^*(M, Z) \otimes \mathcal{R}_G$. In particular, E_n is a free $\mathcal{R}_G$-module. We claim that $\operatorname{rk} E_{n+1} < \operatorname{rk} E_n$. Denote by $\mathcal{Q}_G$ the field of rational functions on $\mathfrak{g}$, i.e., the field of fractions of $\mathcal{R}_G$. The differential d_n induces a differential on $E_n \otimes_{\mathcal{R}_G} \mathcal{Q}_G$, which is again non-zero because E_n is a free R_G-module. It follows that

$$\dim_{\mathcal{Q}_G} H^*(E_n \otimes_{\mathcal{R}_G} \mathcal{Q}_G) < \dim_{\mathcal{Q}_G} E_n \otimes_{\mathcal{R}_G} \mathcal{Q}_G.$$

By the universal coefficients formula,

$$H^*(E_n \otimes_{\mathcal{R}_G} \mathcal{Q}_G) = H^*(E_n) \otimes_{\mathcal{R}_G} \mathcal{Q}_G = E_{n+1} \otimes_{\mathcal{R}_G} \mathcal{Q}_G.$$

By definition, $\operatorname{rk} E_i = \dim_{\mathcal{Q}_G} E_i \otimes_{\mathcal{R}_G} \mathcal{Q}_G$. Therefore,

$$\operatorname{rk} E_{n+1} < \operatorname{rk} E_n.$$

Finally,

$$\mathrm{rk} H^*_G(M,Z) = \mathrm{rk} E_\infty \le \mathrm{rk} E_{n+1} < \mathrm{rk} E_n = \dim H^*(M,Z).$$

□

Among the most interesting examples of formal G-manifolds are compact symplectic manifolds with Hamiltonian G-actions. (In fact, such manifolds are equivariantly perfect, which is a geometric property that implies formality, [**Kir**].) We will see other examples of formal pairs in Appendix G.

The following result gives a useful criterion for formality:

Proposition C.25. *Let G be a compact Lie group. A compact G-manifold M is formal if and only if $H^*_G(M)$ is torsion–free as an $\mathcal{R}_G$-module.*

When G is a torus, this fact is an easy consequence of Borel's localization theorem (see Section 4.2). We defer the proof of the general case to Section 5.3. To check formality, it suffices to only consider torus actions:

Proposition C.26. *Let G be a compact Lie group and $\mathbb{T}$ its maximal torus. A compact G-manifold M is formal if and only if M is formal as a $\mathbb{T}$-manifold.*

We will prove this result in Section 5.3.

4.2. Formality for torus actions. Combining Lemma C.24 with Borel's localization theorem, we obtain the following result:

Corollary C.27. Let G be a torus. A pair of G-manifolds (M,Z) is formal if and only if one of the following conditions holds:

- $\dim H^*(M,Z) = \dim H^*(M^G, Z^G)$;
- $H^*_G(M,Z)$ has no $\mathcal{R}_G$-torsion;
- the restriction $j^*\colon H^*_G(M,Z) \to H^*_G(M^G,Z^G) = H^*(M^G,Z^G)\otimes \mathcal{R}_G$ is a monomorphism.

In general, the geometrical meaning of formality is not completely clear. Let us prove, however, a simple geometric consequence of formality, [**GGK3**]. Recall that an orbit type stratum in M is called *minimal* if it is closed.

Proposition C.28. *Let G be a torus. On a compact formal G-manifold M, every minimal stratum is a connected component of the fixed point set M^G.*

Proof. Let X be a minimal stratum and let H be the connected component of the identity in G_x for $x \in X$. Assume $H \neq G$. Then the equivariant Thom class τ of the normal bundle to X (see, e.g., Section 7) is a non-zero torsion element in $H^*_G(M)$. In fact, τ is annihilated by the image of $H^*(B(G/H)) \to H^*(BG)$. Alternatively, $j^*\tau = 0$, because $X \cap M^G = \emptyset$. □

For example, when M is compact symplectic with G acting Hamiltonianly, M is formal and Proposition C.28 applies. In this case, however, to show that a minimal stratum X consists of fixed points, it suffices to observe that X is a compact symplectic manifold and H acts Hamiltonianly on X, so H must have fixed points on X.

Remark C.29. The condition stated in Proposition C.28 is not sufficient for formality. For instance, let G be a torus and let M be a formal G-manifold. Identify a neighborhood of a free orbit with $G\times D$ where D is a disc. Attach M to $G\times N$, for an interesting compact manifold N, by taking the product of G with the connected sum of D and N. This gives a non-formal manifold with the same minimal strata as in M.

The following result will be used in Appendix G.

LEMMA C.30. *Let G be a torus and let (M, Z) be a pair of G-spaces. Suppose that the pair is formal, and suppose that $Z^G = \emptyset$. Then the restriction map*

$$H_G^*(M) \to H_G^*(Z)$$

is onto.

PROOF. The long exact sequence of the pair (M, Z) gives rise to the sequence

$$H_G^*(M) \to H_G^*(Z) \to H_G^{*+1}(M, Z), \tag{C.11}$$

which is exact in the middle term. Since $Z^G = \emptyset$, by Borel's localization theorem, $H_G^*(Z)$ is an $\mathcal{R}_G$-torsion module. Therefore, its image in $H_G^*(M, Z)$ is also a torsion module. By formality, $H_G^*(M, Z)$ is torsion free, and hence the image of $H_G^*(Z)$ is zero. Since (C.11) is exact and the second map in (C.11) is zero, the first map is onto. □

5. The relation between $\mathbf{H}^*_\mathbf{G}$ and $\mathbf{H}^*_\mathbb{T}$

5.1. The splitting principle. Let G be a compact Lie group, $\mathbb{T}$ a maximal torus in G, and $P \to B$ a principal G-bundle with a compact base B. In this subsection we prove a general result, known as the splitting principle, relating $H^*(B)$ and $H^*(P/\mathbb{T})$. Denote by $\pi \colon P/\mathbb{T} \to B$ the natural projection. Note that W, the Weyl group of G, acts on $P/\mathbb{T}$ in a fiberwise way, and hence on $H^*(P/\mathbb{T})$.

THEOREM C.31 (The splitting principle). *The induced homomorphism*

$$\pi^* \colon H^*(B) \to H^*(P/\mathbb{T})$$

is a monomorphism whose image is exactly $H^*(P/\mathbb{T})^W$.

REMARK C.32. The assertion that π^* is a monomorphism (and its image is in $H^*(P/\mathbb{T})^W$) is relatively straightforward (see Fact 1 below) and can be proved by entirely elementary means. We will refer to this assertion, which is sufficient for many applications, as the *easy part* of the splitting principle. The statement that the image is exactly $H^*(P/\mathbb{T})^W$ is more delicate. Its proof relies heavily on Borel's calculation of $H^*(G/\mathbb{T})$.

It will also be clear from the proof that $H^*(P/\mathbb{T}) = H^*(B) \otimes H^*(G/\mathbb{T})$ as $H^*(B)$-modules. The cohomology of the "flag manifold" $G/\mathbb{T}$ will be described in the proof of Theorem C.31.

REMARK C.33. The references to the original proofs of this theorem and other results of this section can be found in [**Bor1**]. (See also [**DKV**] for the approach using equivariant differential forms.)

EXAMPLE C.34. Let $G = \mathrm{U}(r)$. Consider the vector bundle E over B associated with P, i.e., P is the unitary frame bundle in E. Then the pull-back π^*E over $P/\mathbb{T}$ splits into the direct sum of r complex line bundles $L_1 \oplus \ldots \oplus L_r$. (Hence, the name of the theorem.) Therefore, the total Chern class $c(\pi^*E)$ is the product of Chern classes: $c(\pi^*E) = c(L_1) \cdot \ldots \cdot c(L_r)$. Combined with the fact that π^* is a monomorphism in cohomology, this often allows one to assume in cohomology calculations that a given vector bundle is the direct sum of line bundles. (See, e.g.,

Appendix I where we use the splitting principle to define the Todd classes.) Note also that this version of the splitting principle holds for complex K-equivariant vector bundles over K-manifolds. The formulation of the precise statement and its proof are left to the reader as an exercise.

PROOF OF THEOREM C.31. The proof is based on the following two facts.

Fact 1. *Let* $\pi\colon Q \to B$ *be a fiber bundle with fiber* F *such that the restriction* $H^*(Q) \to H^*(F)$ *is an epimorphism. Then* $H^*(Q) = H^*(B) \otimes H^*(F)$ *as an* $H^*(B)$*-module.* This observation, sometimes called the Leray–Hirsch theorem, is an immediate consequence of the Serre spectral sequence of π. Alternatively, it can be proved by using the Mayer–Vietoris exact sequence for the restriction of π to some nice cover of B (see, e.g., [**Hus**, Chapter 17]). Note that the easy part (monomorphism) of the splitting principle readily follows from this fact.

Fact 2.

$$H^*(G/\mathbb{T}) = S^*\mathfrak{t}^*/(S^{*>0}\mathfrak{t}^*)^W,$$

where $\mathfrak{t}$ *is the Lie algebra of* $\mathbb{T}$*. More explicitly, consider the complex vector bundle over* $G/\mathbb{T}$ *associated with the principal* $\mathbb{T}$*-bundle* $G \to G/\mathbb{T}$ *and the standard representation of* $\mathbb{T} \cong (S^1)^r$ *on* $\mathbb{C}^r$*. This vector bundle decomposes as the sum of complex line bundles* $L_1, \dots, L_r$*. Denote by* $t_1, \dots, t_r$ *the first Chern classes of these line bundles. Then*

$$H^*(G/\mathbb{T}) = \mathbb{R}[t_1, \dots, t_r]/(\mathbb{R}_{\deg>0}[t_1, \dots, t_r])^W,$$

where $\mathbb{R}_{\deg>0}[t_1, \dots, t_r]$ *stands for space of polynomials of positive degree.* This fact is proved by a careful analysis of the Serre spectral sequence of the fiber bundle $B\mathbb{T} \to BG$ whose fiber is $G/\mathbb{T}$; see [**Bor1**].

To prove the theorem, it suffices to note that the vector bundle associated with the principal $\mathbb{T}$-bundle $P \to P/\mathbb{T}$ splits into the sum of line bundles whose restrictions to the fiber are exactly the bundles $L_1, \dots, L_r$. Thus $H^*(P/\mathbb{T}) \to H^*(G/\mathbb{T})$ is an epimorphism, and hence $H^*(P/\mathbb{T}) = H^*(B) \otimes H^*(G/\mathbb{T})$. It follows that π^* is a monomorphism and, since $H^{*>0}(G/\mathbb{T})^W = 0$, the image of π^* is exactly $H^*(P/\mathbb{T})^W$. ☐

5.2. The coefficient ring: $\mathbf{H}^*(\mathbf{BG}) = (\mathbf{S}^*(\mathfrak{g}^*))^{\mathbf{G}}$. As an immediate application of the splitting principle we obtain the identification (C.6), i.e., $H^*(BG) = (S^*(\mathfrak{g}^*))^G$, which is the first step in the proof of the equivariant de Rham theorem (Theorem C.4). Indeed, let $B = BG$ and $P = EG$. Then we have a natural identification $EG/\mathbb{T} = B\mathbb{T}$, and $\pi^*\colon H^*(BG) \to H^*(B\mathbb{T})^W = (S^*\mathfrak{t}^*)^W$ is an isomorphism by the splitting principle. Finally, by Proposition C.12, $(S^*\mathfrak{t}^*)^W$ is naturally isomorphic to $(S^*\mathfrak{g}^*)^G$.

Alternatively, (C.6) can be described more geometrically using the Chern–Weil construction. Then the proof that the resulting map is an isomorphism uses only the easy part of the splitting principle.

Indeed, the Chern–Weil construction (see, e.g., [**BT2, KN**]) gives rise to an algebra homomorphism

$$I\colon (S^*\mathfrak{g}^*)^G \to H^*(BG).$$

Let us show that this homomorphism is in fact an isomorphism. It is not hard to see that the following diagram is commutative:

$$\begin{array}{ccc} (S^*\mathfrak{g}^*)^G & \xrightarrow{\ I\ } & H^*(BG) \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi^*} \\ (S^*\mathfrak{t}^*)^W & \longrightarrow & H^*(B\mathbb{T})^W \end{array}$$

Here the horizontal arrows are given by the Chern–Weil construction. The first vertical arrow is the restriction map from Proposition C.12 and π^* is the splitting homomorphism. It is clear that the bottom horizontal arrow is an isomorphism. By the easy part of the splitting principle, π^* is a monomorphism and by Proposition C.12 the first vertical arrow is an isomorphism. It readily follows from the diagram that I is an isomorphism.

Finally, the identification (C.6) can be proved directly by analyzing the spectral sequence of the fibration $EG \to BG$. Namely, one can show that $H^*(G)$ has generators which are transgressive and that their differentials generate $(S^*\mathfrak{g}^*)^G$.

5.3. $\mathbf{H^*_G(M)}$ and $\mathbf{H^*_T(M)}$. Let, as above, G be a compact Lie group and $\mathbb{T} \subset G$ a maximal torus. In this section we recall how the cohomology groups $H^*_G(M)$ and $H^*_{\mathbb{T}}(M)$, of a G-manifold M, are related to each other.

The inclusion $\mathbb{T} \to G$ gives rise to a homomorphism $H^*_G(M) \to H^*_{\mathbb{T}}(M)$ induced by the mapping

$$(EG \times M)/\mathbb{T} \to (EG \times M)/G,$$

where on the left-hand side we can take EG as $E\mathbb{T}$. Note that, as before, the Weyl group W of G acts on $H^*_{\mathbb{T}}(M)$. For example, when M is a point we have $H^*_{\mathbb{T}}(\mathrm{pt}) = H^*(B\mathbb{T}) = S^*\mathfrak{t}^*$ and $H^*_G(\mathrm{pt}) = H^*(BG) = (S^*\mathfrak{t}^*)^W \hookrightarrow S^*\mathfrak{t}^*$ as discussed in Section 5.2.

Theorem C.35. *Let M be a compact G-manifold.*

(1) $H^*_G(M) \cong H^*_{\mathbb{T}}(M)^W$ *as algebras.*
(2) $H^*_{\mathbb{T}}(M) \cong H^*_G(M) \otimes_{\mathcal{R}_G} \mathcal{R}_{\mathbb{T}}$ *as $\mathcal{R}_{\mathbb{T}}$-modules.*

Proof. The first assertion is just the statement of Theorem C.31 for the principle G-bundle $P = EG \times M \to (EG \times M)/G = B$. Strictly speaking, Theorem C.31 does not apply to this bundle because the base B is not compact or even finite–dimensional. However, B becomes compact when EG is replaced by its finite–dimensional approximation EG_k as in Example C.1. Moreover, this substitute has no effect on the qth cohomology if k is large enough. Now the first assertion does follow from Theorem C.31.

To prove the second assertion, consider the commutative diagram

$$\begin{array}{ccccc} EG \times_{\mathbb{T}} M & \xrightarrow{\ \rho_{\mathbb{T}}\ } & EG/\mathbb{T} & = & B\mathbb{T} \\ {\scriptstyle \pi}\big\downarrow & & & & \big\downarrow \\ EG \times_G M & \xrightarrow{\ \rho_G\ } & EG/G & = & BG \end{array}$$

in which the horizontal arrows are given by the projections to the first component and the vertical arrows are associated bundles with fiber $G/\mathbb{T}$. The structure of the $\mathcal{R}_{\mathbb{T}}$-module on $H^*_{\mathbb{T}}(M)$ is defined by $\rho^*_{\mathbb{T}}$ and the structure of the $\mathcal{R}_G$-module on $H^*_G(M)$ comes from ρ^*_G. Therefore, the $\mathcal{R}_{\mathbb{T}}$-linear homomorphism

$$\varphi\colon H^*_G(M) \otimes_{\mathcal{R}_G} \mathcal{R}_{\mathbb{T}} \to H^*_{\mathbb{T}}(M)$$

which sends $v \otimes u$ to $(\rho_{\mathbb{T}}^* u)(\pi^* v)$ is well defined. Denote by $u_1, \dots, u_r$ the generators of $H^*(B\mathbb{T})$ which are the first Chern classes of the corresponding line bundles. Then $\rho_{\mathbb{T}}^* u_i$ restricts to the class t_i on the fiber $G/\mathbb{T}$ of π for all $i = 1, \dots, r$. It follows from the proof of Theorem C.31 that φ is an isomorphism of graded vector spaces. As a consequence, φ is also an isomorphism of $\mathcal{R}_{\mathbb{T}}$-modules. □

Remark C.36. The second assertion of Theorem C.35 does not follow directly from the Serre spectral sequence for the fibration π because this spectral sequence might, in principle, lose some information about the multiplication on $H_{\mathbb{T}}^*(M)$ and, in particular, about the $\mathcal{R}_{\mathbb{T}}$-module structure on $H_{\mathbb{T}}^*(M)$. A different topological proof of the second assertion can be found in [**Hs**, Section III.1] and an analytical proof in [**DKV**, page 160].

Theorem C.35 is very useful to reduce the analysis of $H_G^*(M)$ to the analysis of $H_{\mathbb{T}}^*(M)$ which is usually simpler. Let us illustrate this by applying Theorem C.35 to prove Propositions C.25 and C.26. More examples will follow in Section 8.

Proof of Proposition C.26. Assume that M is G-formal. Then, by definition, $H_G^*(M)$ is the free $\mathcal{R}_G$-module $H^*(M) \otimes \mathcal{R}_G$. By the second part of Theorem C.35,

$$H_{\mathbb{T}}^*(M) = (H^*(M) \otimes \mathcal{R}_G) \otimes_{\mathcal{R}_G} \mathcal{R}_{\mathbb{T}} = H^*(M) \otimes \mathcal{R}_{\mathbb{T}}.$$

Hence, M is $\mathbb{T}$-formal.

Conversely, assume that M is $\mathbb{T}$-formal. Then $H_{\mathbb{T}}^*(M) = H^*(M) \otimes \mathcal{R}_{\mathbb{T}}$ and the Leray spectral sequence collapses. The W-action on $H_{\mathbb{T}}^*(M)$ induces a W-action on $H^*(M) \otimes \mathcal{R}_{\mathbb{T}}$ which is trivial on the first term and equal to the standard action of W on $\mathcal{R}_{\mathbb{T}}$ on the second. Hence, by the first part of Theorem C.35,

$$H_G^*(M) = H^*(M) \otimes \mathcal{R}_{\mathbb{T}}^W = H^*(M) \otimes \mathcal{R}_G,$$

and M is G-formal. □

Proof of Proposition C.25. Assume that M is formal. Then, by definition, $H_G^*(M)$ is a free $\mathcal{R}_G$-module and, in particular, a torsion–free $\mathcal{R}_G$-module.

Conversely, assume that $H_G^*(M)$ is a torsion–free $\mathcal{R}_G$-module. By Proposition C.26 it suffices to show that M is $\mathbb{T}$-formal, where $\mathbb{T}$ is a maximal torus in G. This, in turn, is equivalent to that $H_{\mathbb{T}}^*(M)$ is torsion–free as an $\mathcal{R}_{\mathbb{T}}$-module by Corollary C.27.

Recall that $\mathcal{R}_{\mathbb{T}}$ is a free $\mathcal{R}_G$-module; see [**Va**, Theorem 4.15.28]. Applying the second part of Theorem C.35, we conclude that $H_{\mathbb{T}}^*(M)$ is torsion–free as an $\mathcal{R}_G$-module, since it is a tensor product of the torsion–free $\mathcal{R}_G$-module $H_G^*(M)$ and the free $\mathcal{R}_G$-module $\mathcal{R}_{\mathbb{T}}$.

Let us show that $H_{\mathbb{T}}^*(M)$ is torsion–free as an $\mathcal{R}_{\mathbb{T}}$-module. Let $u \in H_{\mathbb{T}}^*(M)$ and $f \in \mathcal{R}_{\mathbb{T}}$ be such that $f \neq 0$ and $f \cdot u = 0$. Our goal is to show that $u = 0$. Set $F = \prod_{\gamma \in W} \gamma(f)$. Clearly, $F \in \mathcal{R}_G = \mathcal{R}_{\mathbb{T}}^W$ and $F \neq 0$, since $\mathcal{R}_{\mathbb{T}}$ is the ring of polynomials on $\mathfrak{t}$. Furthermore, $F \cdot u = 0$ and hence u is a torsion element in $H_{\mathbb{T}}^*(M)$ over $\mathcal{R}_G$. As we have shown, $H_{\mathbb{T}}^*(M)$ is torsion–free over $\mathcal{R}_G$. Thus $u = 0$ as required. □

6. Equivariant vector bundles and characteristic classes

In this section we introduce equivariant vector bundles and characteristic classes and "classify" equivariant complex line bundles. We refer the reader to, e.g., [**Hus, MiSt**], for the construction of ordinary (non-equivariant) characteristic classes. Throughout this section, G is a compact Lie group and M is a connected G-manifold.

6.1. Equivariant vector bundles.

DEFINITION C.37. A G-equivariant vector bundle over M is a vector bundle E together with a lift of the G-action to E by fiberwise linear transformations.

EXAMPLE C.38. Let V be a linear representation of G. The vector bundle $E = M \times V$ with the diagonal G-action is a G-equivariant vector bundle. It is a *trivial* G-equivariant vector bundle if the representation of G on V is trivial. When V is a non-trivial representation, E is trivial as an ordinary vector bundle, but not as an equivariant vector bundle.

EXAMPLE C.39. The tangent bundle TM and any tensor bundle associated with TM, e.g., T^*M, are naturally G-equivariant vector bundles over M.

EXAMPLE C.40. The pre-quantization line bundle $\mathbb{L}$ for a Hamiltonian G-action on M is a G-equivariant complex line bundle over M. (See Section 2 of Chapter 6.)

6.2. Equivariant characteristic classes. A G-equivariant vector bundle E on M gives rise to a vector bundle $\tilde{E}$ on $EG \times_G M$ whose pull-back to $EG \times M$ is $EG \times E$. The *equivariant characteristic classes* of E are, by definition, the characteristic classes of $\tilde{E}$. These characteristic classes lie in $H^*_G(M)$. This construction applies to essentially any characteristic class. In particular, one has the equivariant Pontrjagin classes $p^G_k(E)$ when E is any equivariant real vector bundle, the equivariant Euler class $e^G(E)$ when E is an oriented equivariant real vector bundle, and the equivariant Chern classes $c^G_k(E)$ and the Todd class when E is an equivariant complex vector bundle. A slightly different construction of equivariant Chern classes, based on the splitting principle, is outlined in Section 2 of Appendix I.

As follows from the definition, equivariant characteristic classes have many of the properties of ordinary characteristic classes, such as the product formula.

EXAMPLE C.41. Assume that the G-action on M is trivial. Let $\mathbb{L}$ be a G-equivariant complex line bundle over M. (Note that the action of G on $\mathbb{L}$ may still be non-trivial.) Then $H^*_G(M) = H^*(M) \otimes \mathcal{R}_G$, as pointed out in Example C.6. Recall that the curvature class of a line bundle is essentially equal to the cohomology class of the curvature form on the base; this class differs from the first Chern class by a factor, see Appendix A. The equivariant curvature class of $\mathbb{L}$ is

$$[w^G](\mathbb{L}) = [w](\mathbb{L}) - \alpha \in H^*_G(M) = H^*(M) \otimes \mathcal{R}_G,$$

where $[\omega](\mathbb{L})$ is the curvature class of $\mathbb{L}$ and $\alpha \in (\mathfrak{g}^*)^G$ is the weight of the G-representation on a fiber of $\mathbb{L}$, and the equivariant Chern class of $\mathbb{L}$ is

$$c^G_1(\mathbb{L}) = c_1(\mathbb{L}) - \frac{1}{2\pi}\alpha.$$

(See Section 2 of Appendix I and, in particular, Lemma I.3 for further details on this example.)

EXAMPLE C.42. Let $\mathbb{L}$ be a G-equivariant complex line bundle over M. Let $p\colon P \to M$ be the unit circle bundle in $\mathbb{L}$ with respect to some G-invariant fiberwise Hermitian metric. Pick a G-invariant connection form Θ on P. Then $c^G_1(\mathbb{L}) =$

$\frac{1}{2\pi}[\omega - \Phi]$ in $H^2_G(M)$, where ω is the curvature form on M, i.e., $p^*\omega = -d\Theta$ and the moment map is defined by the condition $\pi^*\Phi = \Theta(\xi_P)$. In other words,

$$\frac{1}{2\pi}p^*(\omega - \Phi) = -d_G\Theta. \tag{C.12}$$

(For more details on these examples, see Section 2.4 of Chapter 6 and Appendix A.)

This definition of c_1^G is a simple example of the equivariant Chern–Weil construction which can also be carried out to define other equivariant characteristic classes; see [**BT2, BV1, BV2**] and [**BGV**].

EXAMPLE C.43. For a G-manifold M, the equivariant Euler class $e^G(M)$ and the equivariant Pontrjagin classes $p_k^G(M)$ are, by definition, the equivariant Euler and Pontrjagin classes of TM. Similarly, if M carries a G-invariant almost complex structure or a G-equivariant stable complex structure, one has the equivariant Chern classes $c_k^G(M)$.

EXAMPLE C.44. The normal bundle $\mathcal{V}$ to the fixed point set M^G is a G-equivariant vector bundle. Thus the equivariant characteristic classes of $\mathcal{V}$ are defined. Since the G-action on M^G is trivial, these classes belong to $H^*_G(M) \otimes \mathcal{R}_G$, where, as above, $\mathcal{R}_G$ is the space of G-invariant polynomials on $\mathfrak{g}$.

The Euler class $e^G(\mathcal{V})$ will play a particularly important role in Section 7. Let us assume that G is a torus and obtain an expression for the leading term of this class as a polynomial on $\mathfrak{g}$. To this end, we will restrict our attention to one connected component F of M^G. Its normal bundle $\mathcal{V}_F$ can be turned into a *complex* G-equivariant vector bundle. Indeed, choose a subcircle $S^1 \subseteq G$ that acts with the same fixed points as G. Let $\mathcal{V}_F = \bigoplus \mathcal{V}_m$ be the splitting such that S^1 acts on $\mathcal{V}_m$ with stabilizer $\{e^{2\pi ik/m} \mid 0 \leq k < m\}$. Define the multiplication by $\sqrt{-1}$ to be

$$Jv = e^{2\pi i/4m} \cdot v$$

for $v \in \mathcal{V}_m$. Let $\alpha_1, \dots, \alpha_k$, be the weights of the G-representation on a fiber of $\mathcal{V}_F$. Then

$$e^G(\mathcal{V}_F) = (-1)^k \prod_{i=1}^{k} \alpha_i + \dots \tag{C.13}$$

where the dots stand for lower degree terms. This formula follows from Example C.41 and the splitting principle and is used in Section 7. Since F is a component of M^G, all weights α_i are non-zero and, as a consequence, so is the leading term. In particular, $e^G(\mathcal{V}_F)$ is a polynomial of degree $\operatorname{codim} F/2$.

Note in conclusion that whereas the weights α_i depend on the choice of complex structure, their product depends only on the choice of orientation. Indeed, as in the non-equivariant case, the Euler class is defined for an *oriented* vector bundle.

Also note that to determine the leading term of $e^G(\mathcal{V})$ it suffices to consider the representation of G on a single fiber.

REMARK C.45. In the definition of equivariant characteristic classes we used the fact that a G-equivariant vector bundle E over M gives rise to a vector bundle $\tilde{E}$ over $EG \times_G M$. This construction leads to the question of whether or not the correspondence $E \mapsto \tilde{E}$ is a bijection between the sets of isomorphism classes of equivariant vector bundles on M and ordinary vector bundles over $EG \times_G M$, which we denote by $\mathrm{Vect}^G(M)$ and $\mathrm{Vect}(EG \times_G M)$, respectively. This turns out to be a very subtle question even when $M = \mathrm{pt}$. For example, recall that $\mathrm{Vect}^G(\mathrm{pt})$ is the set of isomorphism classes of representations of G and $\mathrm{Vect}(EG \times_G \mathrm{pt}) = \mathrm{Vect}(BG)$. When G is a torus, $\mathrm{Vect}^G(\mathrm{pt}) \to \mathrm{Vect}(BG)$ is a bijection. Furthermore, when G is connected, this map induces an isomorphism of the corresponding Grothendieck groups. If G is not connected, the map $\mathrm{Vect}^G(\mathrm{pt}) \to \mathrm{Vect}(BG)$ may fail to be a bijection. The reader interested in the references to these and other related results should consult the survey [**Ol**] and also [**JMcCO**].

The situation considerably simplifies (at least for continuous G-actions) for complex line bundles or more generally for vector bundles with structural group $\mathrm{U}(1)^n$. Namely, in this case there is a one-to-one correspondence between (continuous) G-equivariant bundles and bundles over $EG \times_G M$; see [**HY**].

6.3. Equivariant complex line bundles. In this section we classify G-equivariant line bundles. Recall that, as is well known, ordinary complex line bundles are classified by the first Chern class:

PROPOSITION C.46. *The first Chern class establishes a one-to-one correspondence between the collection of isomorphism classes of complex line bundles on M and $H^2(M;\mathbb{Z})$.*

PROOF. Fix a cover U_i of M by contractible open sets. A complex line bundle $\mathbb{L} \to M$ trivializes over each U_i. Its transition functions form a Čech one-cocycle $\varphi_{ij}\colon U_i \cap U_j \to \mathbb{C}^\times$. Two cocycles, φ_{ij} and ψ_{ij}, correspond to isomorphic line bundles if and only if they are in the same cohomology class: $\varphi_{ij} = f_i \psi_{ij} f_j^{-1}$ for some $f_j\colon U_j \to \mathbb{C}^\times$. Hence, line bundles are classified by $\check{H}^1(M;\mathbb{C}^\times)$. This group is isomorphic to $H^2(M; 2\pi\mathbb{Z})$. This follows from the long exact sequence in cohomology coming from the short exact sequence of sheaves $0 \to 2\pi\mathbb{Z} \to \mathbb{C} \stackrel{\exp}{\to} \mathbb{C}^\times \to 1$, combined with the fact that the sheaf of smooth $\mathbb{C}$-valued functions is flabby. Dividing by 2π, we conclude that the line bundles $\mathbb{L}$ are classified by the resulting elements $c_1(\mathbb{L}) \in H^2(M;\mathbb{Z})$. For more details we refer the reader to, e.g., [**Ki4, Kost**].

Alternatively, the proposition is a consequence of the identification $K(\mathbb{Z},2) = \mathbb{CP}^\infty = B\,\mathrm{U}(1)$. □

This correspondence between line bundles and integral cohomology classes was used, for example, in Section 2.2 of Chapter 6 to determine which closed two-forms admit a pre-quantization. Our goal now is to extend this result to equivariant line bundles. This result is of interest for us for two reasons: it gives a necessary and sufficient condition for a Hamiltonian G-manifold to be pre-quantizable (Theorem 6.7), and it is also used in the classification of equivariant $\mathrm{Spin}^{\mathbb{C}}$-structures (Section 2 of Appendix D).

THEOREM C.47. *Let G be a compact Lie group and M a connected G-manifold. The equivariant first Chern class c_1^G gives rise to a one-to-one correspondence between equivalence classes of G-equivariant complex line bundles over M and elements of $H^2_G(M;\mathbb{Z})$.*

REMARK C.48. For locally finite CW-complexes with continuous G-actions, this result is proved by Hattori and Yoshida, [**HY**]. For smooth actions of connected groups, Theorem C.47 is due to Riera, [**Ri**]. The proof given below is different from either of these proofs.

REMARK C.49. Let Γ be the set of isomorphism classes of G-equivariant complex line bundles over M. It is easy to see that Γ is a group with multiplication given by tensor product of line bundles, and that $c_1^G\colon \Gamma \to H^2_G(M;\mathbb{Z})$ is a group homomorphism. According to Theorem C.47, this is in fact an isomorphism.

PROOF OF THEOREM C.47. Consider a finite–dimensional approximation

$$EG_k \to EG_k/G = BG_k$$

of the universal bundle $EG \to BG$, as in Example C.1, with the following properties:

- EG_k is a smooth manifold with a free G-action;
- EG_k is simply connected and $H^2(EG_k;\mathbb{Z}) = 0$;
- the inclusion
$$j\colon M \times_G EG_k \hookrightarrow EG \times_G M$$
induces an isomorphism
$$j^*\colon H^2_G(M;\mathbb{Z}) = H^2(EG \times_G M;\mathbb{Z}) \to H^2(EG_k \times_G M;\mathbb{Z}).$$

The actual construction of such an approximation is immaterial for the argument below and we only need the fact that this approximation exists. See Example C.1 for a construction of such an approximation.

Recall that to every G-equivariant line bundle $\mathbb{F}$ over M one can associate an ordinary line bundle $\mathbb{L}$ over $EG_k \times_G M$ as follows. Let $\pi\colon EG_k \times M \to M$ be the natural projection. The pull-back $\pi^*\mathbb{F}$ is G-equivariant with respect to the diagonal G-action on $EG_k \times M$. Since the diagonal action is free, $\pi^*\mathbb{F}$ descends to a line bundle $\mathbb{L} = \mathbb{F}/G$ over $EG_k \times_G M$. In other words, $\mathbb{L}$ is characterized by the condition

$$\tilde{\mathbb{L}} = \pi^*\mathbb{F}, \tag{C.14}$$

where $\tilde{\mathbb{L}}$ is the pull-back of $\mathbb{L}$ to $EG_k \times M$. □

LEMMA C.50.

(1) $j^*c_1^G(\mathbb{F}) = c_1(\mathbb{L})$.
(2) *The correspondence* $\mathbb{F} \mapsto \mathbb{L}$ *gives rise to a bijection between* G*-equivariant complex line bundles* $\mathbb{F}$ *over* M *and ordinary complex line bundles* $\mathbb{L}$ *over* $EG_k \times_G M$*, up to isomorphism.*

The theorem follows from the lemma combined with Proposition C.46 and the fact that j^* is an isomorphism in degree two.

PROOF OF THE LEMMA. Let us prove that $j^*c_1^G(\mathbb{F}) = c_1(\mathbb{L})$. First note that replacing EG_k by EG in the definition of $\mathbb{L}$, we obtain a line bundle $\mathbb{L}'$ over $EG \times_G M$ for a G-equivariant line bundle $\mathbb{F}$. (This is exactly the construction described in Section 6.2.) The diagram

$$\begin{array}{ccccccc}
\mathbb{L} & \longrightarrow & EG_k \times_G M & \overset{j}{\hookrightarrow} & EG \times_G M & \longleftarrow & \mathbb{L}' \\
 & & \uparrow & & \uparrow & & \\
\tilde{L}{=}\pi^*\mathbb{F} & \longrightarrow & EG_k \times M & \hookrightarrow & EG \times M & & \\
 & & {\scriptstyle\pi}\downarrow & & \downarrow & & \\
\mathbb{F} & \longrightarrow & M & = & M & \longleftarrow & \mathbb{F}
\end{array}$$

is commutative, and, hence, $\mathbb{L} = j^*\mathbb{L}'$. By definition, $c_1^G(\mathbb{F}) = c_1(\mathbb{L}')$ and, therefore,

$$j^* c_1^G(\mathbb{F}) = j^* c_1(\mathbb{L}') = c_1(j^*\mathbb{L}') = c_1(\mathbb{L}).$$

Let us prove that the correspondence $\mathbb{F} \mapsto \mathbb{L}$ is surjective. Let $\mathbb{L}$ be a complex line bundle over $EG_k \times_G M$. As above, denote by $\tilde{\mathbb{L}}$ the pull-back of $\mathbb{L}$ to $EG_k \times M$. By (C.14), we need to find a G-equivariant line bundle $\mathbb{F}$ over M such that $\pi^*\mathbb{F} = \tilde{\mathbb{L}}$.

Since $H^2(EG_k; \mathbb{Z}) = 0$, every line bundle over EG_k is trivial. In particular, the restriction of $\tilde{\mathbb{L}}$ to every fiber of the projection π is trivial. As a consequence, $\tilde{\mathbb{L}}$ admits a fiberwise flat connection, i.e., a connection whose restriction to every fiber of π is flat. Indeed, such a connection exists locally in M, i.e., over sets $EG_k \times U$, where U are contractible open subsets of M, because $\tilde{\mathbb{L}}|_{EG_k \times U}$ is trivial. Using a partition of unity in M subordinate to a locally finite cover of M by such open sets U, we can patch these connections together into a fiberwise flat connection.

Furthermore, the fiberwise connection can be made G-invariant by averaging over the diagonal G-action on $EG_k \times M$. More precisely, let P be the U(1)-principal bundle for $\tilde{\mathbb{L}}$ and Θ_0 the connection form on P for a fiberwise flat connection. Then, since the G-action on $EG_k \times M$ preserves the fibers of π, the connections $g^*\Theta_0$ are also fiberwise flat and so is the connection $\Theta = \int_G g^*\Theta_0 \, dg$; cf. Remark C.51. (Note that this argument uses the fact that U(1) is commutative.) Furthermore, $\pi_1(EG_k) = 0$ implies that the connection Θ is π-fiberwise trivial, i.e, has trivial holonomy, or, in other words, is globally flat.

Now let $\mathbb{F}$ be the complex line bundle over M whose fiber over $x \in M$ is the set of flat sections of $\tilde{\mathbb{L}}$ over $\pi^{-1}(x)$ with respect to Θ. Using the trivialization over $EG_k \times U$, it is easy to show that $\mathbb{F}$ is really a line bundle. Clearly, $\pi^*\mathbb{F} = \tilde{\mathbb{L}}$. Furthermore, $\mathbb{F}$ is G-equivariant, since Θ is G-invariant. This completes the proof of surjectivity.

Let us prove that $\mathbb{F} \mapsto \mathbb{L}$ is injective. Assume that $\mathbb{L}$ is trivial. Then $\tilde{\mathbb{L}} = \pi^*\mathbb{F}$ admits a G-invariant globally flat connection. Denote by $p\colon P \to EG_k \times M$ the U(1)-principal bundle and by Θ the corresponding flat G-invariant connection form on P. This connection, however, need not project to M, for Θ may not agree with the connection Θ_π along the fibers of π, which $\pi^*\mathbb{F}$ acquires as a pull-back bundle.

We will modify Θ to obtain a connection that does project to a globally flat connection on M. Note that Θ_π is G-invariant and that $\Theta - \Theta_\pi = p^*\alpha$, where α is a closed G-invariant one-form defined along the π-fibers. Since the fibers are simply connected, α is in fact exact on every fiber. Hence, on every fiber, there exists a function f such that $df = \alpha$. This function is defined up to a constant. Let us then fix a section $\{q\} \times M$, where $q \in EG_k$, and choose the function f on every fiber so that $f(q) = 0$. These functions fit together to form a smooth function, denoted again by f, on $EG_k \times M$, such that $d_\pi f = \alpha$, where d_π is the de Rham differential along the fibers of π. Finally, let $\bar{f}$ be the average $\bar{f} = \int_G g^* f \, dg$. Then $\bar{f}$ is G-invariant and $d_\pi \bar{f} = \alpha$, for α is also G-invariant. The connection $\Theta' = \Theta_0 - d\bar{f}$ is G-invariant, globally flat on $EG_k \times M$, and such that $\Theta' = \Theta_\pi$ on the fibers of π.

It remains to show that Θ' projects to a globally flat connection on M. (Note that the projection connection will automatically be G-invariant if it exists.) For a path γ in M connecting points x and y, consider its arbitrary lift $\tilde{\gamma}$ to $EG_k \times M$, connecting points $\tilde{x}$ and $\tilde{y}$. We have natural identifications $(\pi^*\mathbb{F})_{\tilde{x}} = \mathbb{F}_x$ and $(\pi^*\mathbb{F})_{\tilde{y}} = \mathbb{F}_y$. For the projection connection, the parallel transport φ from $\mathbb{F}_x$ to $\mathbb{F}_y$ along γ is then defined as the parallel transport $\tilde{\varphi}$ from $(\pi^*\mathbb{F})_{\tilde{x}}$ to $(\pi^*\mathbb{F})_{\tilde{y}}$ along $\tilde{\gamma}$.

Since the connection on $EG_k \times M$ is globally flat, $\tilde{\varphi}$ is independent of the lift $\tilde{\gamma}$ as long as $\tilde{x}$ and $\tilde{y}$ are fixed. Furthermore, since $\Theta' = \Theta_\pi$, it follows that φ is actually independent of $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{y} \in \pi^{-1}(y)$ and of the path connecting x and y. This establishes injectivity and completes the proof of Lemma C.50 and Theorem C.47. □

REMARK C.51. Let $\mathbb{L}$ be a G-equivariant line bundle over M which admits a globally flat connection, i.e., a connection with trivial holonomy. Then $\mathbb{L}$ also admits a G-invariant connection which is locally flat, i.e., has zero curvature.

The proof is by averaging. Let $p\colon P \to M$ be the U(1)-principal bundle for $\mathbb{L}$ and Θ_0 a connection form on P with trivial holonomy. For every $g \in G$, the connection form $g^*\Theta_0$ has zero curvature. Thus $g^*\Theta_0 = \Theta_0 + p^*\alpha_g$, where α_g is a closed one-form on M. It follows that the G-invariant connection

$$\Theta = \int_G g^*\Theta_0 \, dg = \Theta_0 + p^* \int_G \alpha_g \, dg$$

also has zero curvature. (Here the Haar measure dg is normalized so that G has unit volume.)

In general, Θ need not be globally flat. However, this is always the case when G is connected. Indeed, the form α_g depends smoothly on g and its cohomology class is integral. Hence, for a connected group G, the form α_g is exact for all g.

As a consequence of the last remark, the proof that $\mathbb{F} \mapsto \mathbb{L}$ is injective can be considerably simplified when G is connected: it suffices first to pull-back a globally flat connection on $\pi^*\mathbb{F}$ to a globally flat connection on $\mathbb{F}$ by any section $M \to EG_k \times M$. The resulting pull-back connection can then be made G-invariant, while still kept globally flat, by averaging over G.

7. The Atiyah–Bott–Berline–Vergne localization formula

In this section we will prove a localization formula expressing the integral of a closed equivariant form over a G-manifold through integrals over the fixed point set. This formula was obtained in [**BV1, BV2**] and [**AB2**]. There are different proofs of the localization formula known today. For example, it can be proved using cobordism techniques. Here, however, we mainly follow the topological approach of Atiyah and Bott, [**AB2**].

Throughout this section we assume that G is a *connected* compact Lie group.

7.1. The localization formula. Let β be a compactly supported equivariant differential form of degree q on an oriented G-manifold M. Denote by β_k the component of β in $(\Omega^k(M) \otimes S^*(\mathfrak{g}^*))^G$. For example, β_0 is a smooth function on M taking values in polynomials on $\mathfrak{g}$. More generally, interpreting β_k as a polynomial valued k-form, we set

$$\int_M \beta = \int_M \beta_n \in \mathcal{R}_G,$$

where $n = \dim M$. Note that, by (C.3), this integral may be non-zero even when $\deg \beta \neq \dim M$ in contrast with ordinary integration of differential forms.

It is not hard to see that the equivariant analogue of Stokes' formula holds:

$$\int_M d_G\beta = \int_{\partial M} \beta.$$

Assume now that $\partial M = \emptyset$ and, for the sake of simplicity, that M is compact. Then, by Stokes' formula, the integral over M is a well–defined $\mathcal{R}_G$-linear map

$$\int_M : H_G^*(M) \to \mathcal{R}_G. \tag{C.15}$$

Note that this map is *not* an algebra homomorphism.

Example C.52. An *equivariant characteristic number* of M is the integral over M of an equivariant characteristic class. Note that this "number" is an element of $\mathcal{R}_G$.

It is worth noticing that, since the range of $\int_M$ is torsion–free, the integration map (C.15) kills the torsion part of $H_G^*(M)$. In a similar fashion, integration gives rise to a $\mathcal{Q}_G$-linear homomorphism

$$\int_M : H_G^*(M) \otimes \mathcal{Q}_G \to \mathcal{Q}_G, \tag{C.16}$$

where $\mathcal{Q}_G$ is the field of G-invariant rational functions on $\mathfrak{g}$.

The localization formula expresses the integral over M through the integrals over the components of M^G, when G is torus. This formula, which does not have a non-equivariant analogue, is yet another incarnation of the general principle that for an abelian G many properties of the equivariant cohomology can be recovered from the fixed point set. (See Section 3.)

To state the formula, denote by $\mathcal{V}_F$ the normal bundle to a connected component F of M^G. Fix an orientation of M and a fiberwise orientation of $\mathcal{V}_F$; these determine an orientation of F. By Example C.44 (in particular, (C.13)), the Euler class $e^G(\mathcal{V}_F)$ is invertible in the algebra of $H^*(F)$-valued rational functions on $\mathfrak{g}$, when G is a torus. In contrast, the non-equivariant Euler class is never invertible.

Theorem C.53 (A-B-B-V localization theorem, I). *Let G be a torus acting on compact manifold M. For any class $u \in H_G^*(M)$,*

$$\int_M u = \sum_F \int_F \frac{u|_F}{e^G(\mathcal{V}_F)}, \tag{C.17}$$

where $u|_F$ is the restriction of u to F.

This formula takes a particularly simple form when the fixed points are isolated:

$$\int_M u = (-2\pi)^n \sum_{p \in M^G} \frac{u(p)}{\prod \alpha_{i,p}}, \tag{C.18}$$

where $n = \frac{1}{2} \dim M$ and where $\alpha_{i,p}$ are the isotropy weights for the G-action on T_pM (with the normalization convention of Appendix A).

Remark C.54. Note that every term on the right-hand side of (C.17) is, in general, an element of $\mathcal{Q}_G$, while their sum is an element of $\mathcal{R}_G$.

The localization theorem holds as stated when u is an element of $H_G^*(M) \otimes \mathcal{Q}_G$. Moreover, one can further extend the polynomial ring. For example, the formula remains correct when in the definition of equivariant cohomology the polynomial ring on $\mathfrak{g}$ is replaced by formal power series or analytic functions with values in $\Omega^*(M)$. (See Remark C.14.)

Similar formulas hold, of course, for closed equivariant differential forms (rather than cohomology classes) with an appropriate differential form replacing the cohomology class $e^G(\mathcal{V})$.

EXAMPLE C.55. Let G be a circle and β an equivariantly closed equivariant form. Identifying $\mathfrak{g}$ with $\mathbb{R}$, we can write $\beta = \beta_1 + \ldots + \beta_{\dim M}$ where β_k is a k-form on M with values in the polynomials in an independent variable t. Furthermore, the weights α_i then take the form $\alpha_i = w_i t$, where $w_i \in \mathbb{Z}$. Thus (C.18) becomes

$$\int_M \beta = \left(-\frac{2\pi}{t}\right)^n \sum_{p \in M^G} \frac{\beta_0(p)(t)}{\prod w_{i,p}}.$$

REMARK C.56. Recall that equivariant differential forms on M can be thought of as G-invariant polynomials on $\mathfrak{g}$ with values in $\Omega^*(M)$. For every $\xi \in \mathfrak{g}$ we have the evaluation homomorphism $\Omega^*_G(M) \to \Omega^*(M)$ which sends an equivariant differential form α to an ordinary differential form $\alpha(\xi)$. This homomorphism, in general, does not commute with the differential (if $\xi \neq 0$) and, hence, does not induce a homomorphism in cohomology. Moreover, the form $\alpha(\xi)$ may not even be closed when α is exact.

However, it is easy to see that $\int_M (d_G\beta)(\xi) = 0$ for any form β when M is compact. Hence, for a closed form α, we have

$$\left(\int_M [\alpha]\right)(\xi) = \int_M \alpha(\xi).$$

For this reason the value of this integral is sometimes written as $\int_M [\alpha](\xi)$; see, e.g., Remark I.6.

The evaluation at $\xi = 0$ gives rise to the forgetful algebra homomorphism $H^*_G(M) \to H^*(M)$ which is induced by the restriction to the fiber of the fibration $EG \times_G M \to BG$.

Recall from Example C.6 that when the G-action on M is trivial, $H^*_G(M)$ is canonically identified with $H^*(M) \otimes \mathcal{R}_G$. In this case the evaluation at every ξ is a well–defined homomorphism $H^*_G(M) \to H^*(M)$. In general, when the action is not trivial, the evaluation at $\xi \neq 0$ as a map in cohomology is not well defined.

With this remark in mind we are now in a position to state a slightly more general version of Theorem C.53, which is due to Berline and Vergne, [**BV1**]. This theorem can be proved similarly to Theorem C.53.

THEOREM C.57 (A-B-B-V localization theorem, II). *Let α be a closed equivariant differential form on M. For every $\xi \in \mathfrak{g}$,*

$$\int_M \alpha(\xi) = \sum_F \int_F \frac{\alpha|_F(\xi)}{e^G(\mathcal{V}_F)(\xi)}, \tag{C.19}$$

where F runs through the connected components of $M^\xi = \{\xi_M = 0\}$ and $e^G(\mathcal{V}_F)$ is an equivariant differential form representing the equivariant Euler class of $\mathcal{V}_F$.

7.2. The proof of the localization theorem. Following [**AB2**], let us first recall some properties of the push-forward homomorphism in ordinary cohomology. Let

$$f\colon N \to M$$

be a continuous map of compact oriented manifolds of dimensions n and m, respectively. The *push-forward*

$$f_*\colon H^*(N) \to H^{*-n+m}(M)$$

is defined as the composition

$$f_*\colon H^*(N) \overset{D_N}{\to} H_{n-*}(N) \overset{f_*}{\to} H_{n-*}(M) \overset{D_M^{-1}}{\to} H^{*+m-n}(M), \tag{C.20}$$

where the first and the last arrows are the Poincaré duality homomorphisms and the middle arrow is the induced map in homology. Since we work with cohomology with real coefficients, $f_*v \in H^*(M)$, for $v \in H^*(N)$, is the unique cohomology class such that

$$\int_M (f_*v)x = \int_N v(f^*x) \tag{C.21}$$

for all $x \in H^*(M)$. Equivalently, f_* is dual to f^* with respect to the natural inner product on cohomology: $\langle x, y\rangle = \int xy$.

Example C.58. When $M = \mathrm{pt}$, the push-forward $H^*(N) \to \mathbb{R}$ is given by the integral over N. Furthermore, if f is a fibration, f_* is the integration over the fibers of f.

Let us now list the properties of the push-forward homomorphism which are important for what follows.

Proposition C.59 (The properties of push-forward).

(1) *The push-forward is functorial:* $(fg)_* = f_*g_*$.
(2) *The push-forward of* $f\colon N \to M$ *is linear over* $H^*(M)$. *In other words,* $f_*(vf^*(u)) = f_*(v)u$ *for any* $u \in H^*(M)$ *and* $v \in H^*(N)$.
(3) *Let* $j\colon N \hookrightarrow M$ *be an embedding. Denote by* $E \to N$ *a tubular neighborhood of* N *and let* $k = \operatorname{codim} N$. *Then* j_* *factors into the following sequence of homomorphisms:*

$$H^{*-k}(N) \overset{\mathrm{Th}}{\to} H^*_c(E) \to H^*_c(M) = H^*(M),$$

where the first arrow Th *is the Thom isomorphism (see* [**BT1**]*).*
(4) *In the setting of the previous statement,* $j^*j_*(v) = v\, e(\mathcal{V}_N)$, *where* $\mathcal{V}_N$ *is the normal bundle to* N *in* M.

Let us give some hints on the proofs of these facts. The first property is an immediate consequence of the definition (C.20). The characterization of f_*v given by (C.21) implies the second property. The third property follows from the fact that the image in $H^*(M)$ of the Thom class $\mathrm{Th}(1)$ of $\mathcal{V}_N$ is Poincaré dual to N and that $\mathrm{Th}(v) = \mathrm{Th}(1)\pi^*v$, where π is the projection $E \to N$. (Note also that the third assertion is equivalent to that j_* can be factored as

$$H^{*-k}(N) \overset{\mathrm{Th}}{\to} H^*(E, E\smallsetminus N) \to H^*(M, M\smallsetminus N) \to H^*(M),$$

where the second arrow is the excision homomorphism and the third one is induced by the inclusion.) The fourth assertion is a consequence of the third one and the definition of the Euler class: $j^*\,\mathrm{Th}(1) = e(\mathcal{V}_N)$. We leave detailed proofs to the reader as an exercise. Alternatively, the proofs can be found, for example, in [**Dol2**, Chapter 8].

Remark C.60. As a side remark, let us point out the following useful application of the push-forward. Let N and M be compact manifolds of equal dimensions and $f\colon N \to M$ a continuous mapping of non-zero degree. Then

$f^*\colon H^*(M) \to H^*(N)$ is a monomorphism. Indeed, by the second property of the push-forward,

$$f_* f^* u = f_*(1)u = (\deg f)u.$$

Thus, $(\deg f)^{-1} f_*$ is a left inverse of f^*.

For example, let N and M be closed surfaces of genera k and l, respectively. Then a non-zero degree map $N \to M$ exists if and only if $k \geq l$. (If N or M is not orientable, one should use the push-forward in cohomology with $\mathbb{Z}_2$-coefficients.)

The push-forward is also defined in the equivariant setting. Namely, a G-equivariant mapping $f\colon N \to M$ of compact oriented G-manifolds gives rise to the push-forward $\mathcal{R}_G$-linear homomorphism

$$f_*\colon H_G^*(N) \to H_G^{*-n+m}(M).$$

This homomorphism has the properties 1–4 of Proposition C.59, with all the cohomology spaces and classes replaced by their equivariant counterparts.

To define the *equivariant push-forward*, the ordinary push-forward construction cannot be applied to the map $EG \times_G N \to EG \times_G M$, for these spaces are infinite–dimensional and the Poincaré duality homomorphism is not defined. However, the Poincaré duality is defined when the space EG is replaced by its finite–dimensional approximation EG_k as in Example C.1. Thus, for every q we have the push-forward homomorphism $H^q(EG_k \times_G N) \to H^{q-n+m}(EG_k \times_G N)$. When k is large enough these cohomology spaces are equal to the equivariant cohomology of N and M, respectively, and the push-forward is independent of k. The resulting homomorphism is called the push-forward in the equivariant cohomology. The equivariant version of Proposition C.59 is now relatively straightforward to verify.

Example C.61. Let $\pi\colon N \to \mathrm{pt}$ be the constant map. Then $\pi_* = \int_N$. Furthermore, π_* can be understood as the integration over the fibers of the fibration $EG \times_G N \to BG$.

It is also clear that the push-forward extends to a $\mathcal{Q}_G$-linear homomorphism $H_G^*(N) \otimes \mathcal{Q}_G \to H_G^*(M) \otimes \mathcal{Q}_G$. Thus, to prove the localization theorem, we only need to show that

$$\pi_*^M u = \sum_F \pi_*^F \left(\frac{(j^F)^* u}{e^G(\mathcal{V}_F)} \right), \tag{C.22}$$

where $\pi^M\colon M \to \mathrm{pt}$ and $\pi^F\colon F \to \mathrm{pt}$ are constant maps and $j^F\colon F \hookrightarrow M$ is the natural inclusion. Consider the embedding

$$j\colon M^G \hookrightarrow M.$$

Lemma C.62. *The push-forward*

$$j_*\colon H_G^*(M^G) \otimes \mathcal{Q}_G \to H_G^*(M) \otimes \mathcal{Q}_G$$

is an isomorphism, and its inverse is $u \mapsto \sum (j^F)^* u / e^G(\mathcal{V}_F)$. *In particular, for every* $u \in H_G^*(M)$,

$$j_* \sum_F \frac{(j^F)^* u}{e^G(\mathcal{V}_F)} \equiv u \mod \mathcal{R}_G\text{-}torsion. \tag{C.23}$$

Remark C.63. This lemma does not have a non-equivariant analogue because the Euler class in the ordinary cohomology is never invertible.

Assuming the lemma, let us prove the localization theorem in the form (C.22). Note that $\pi_*^M j_* = \pi_*^{M^G}$ and $\pi_*^{M^G} = \sum_F \pi_*^F$. Then, by the lemma,

$$\begin{aligned} \pi_*^M u &= \pi_*^M j_* \sum_F \frac{(j^F)^* u}{e^G(\mathcal{V}_F)} \\ &= \pi_*^{M^G} \sum_F \frac{(j^F)^* u}{e^G(\mathcal{V}_F)} \\ &= \sum_F \pi_*^F \left(\frac{(j^F)^* u}{e^G(\mathcal{V}_F)} \right). \end{aligned}$$

PROOF OF LEMMA C.62. Since, by Borel's localization, j^* is an isomorphism modulo torsion, to prove (C.23), it suffices to show that

$$j^* j_* \sum_F \frac{(j^F)^* u}{e^G(\mathcal{V}_F)} = j^* u.$$

Furthermore, by the the fourth property of the push-forward, $(j^F)^* j_*^F v = v\, e^G(\mathcal{V}_F)$. Thus,

$$\begin{aligned} j^* j_* \sum_F \frac{(j^F)^* u}{e^G(\mathcal{V}_F)} &= \sum_F (j^F)^* j_*^F \left(\frac{(j^F)^* u}{e^G(\mathcal{V}_F)} \right) \\ &= \sum_F \frac{(j^F)^* u}{e^G(\mathcal{V}_F)} e^G(\mathcal{V}_F) \\ &= j^* u. \end{aligned}$$

This proves (C.23).

By (C.23), the homomorphism $u \mapsto \sum (j^F)^* u / e^G(\mathcal{V}_F)$ is a right inverse of $j_* \colon H_G^*(M^G) \otimes \mathcal{Q}_G \to H_G^*(M) \otimes \mathcal{Q}_G$. By Borel's localization, the $\mathcal{Q}_G$-vector spaces $H_G^*(M^G) \otimes \mathcal{Q}_G$ and $H_G^*(M) \otimes \mathcal{Q}_G$ have equal dimensions. It follows that j_* is an isomorphism. ☐

REMARK C.64. The reader interested in a refinement of this argument, giving some information on the torsion of $H_G^*(M)$, should consult, e.g., [**AB2**, Section 5].

8. Applications of the Atiyah–Bott–Berline–Vergne localization formula

8.1. The Duistermaat–Heckman formula and the relations among characteristic numbers. The localization theorem has numerous applications. One of them is a short proof of the Duistermaat–Heckman formula (see Section 6 of Chapter 4) originally obtained in [**DH1**] by a different method. In its simplest form, when the fixed points are isolated, the Duistermaat–Heckman formula reads

$$\text{(C.24)} \qquad \frac{1}{n!} \int_M e^{\langle \Phi, \xi \rangle} \omega^n = (2\pi)^n \sum_{p \in M^G} \frac{e^{\langle \Phi(p), \xi \rangle}}{\prod \langle \alpha_{i,p}, \xi \rangle}$$

for any equivariantly closed equivariant two-form $\omega - \Phi$ and any vector $\xi \in \mathfrak{g}$ such that $\{\xi_M = 0\} = M^G$, where $\alpha_{i,p}$ are the isotropy weights at p. The Duistermaat–Heckman formula (C.24) follows from the integral localization theorem by applying (C.18) to $u = e^{-(\omega - \Phi)}$ and plugging in $\xi \in \mathfrak{g}$. The condition $\{\xi_M = 0\} = M^G$

guarantees that the denominators on the right-hand side are non-zero. (For more details see [**BV1, BV2, AB2**].)

Another interesting class of results is obtained by applying the localization formula to the characteristic classes of M. This yields relations among the equivariant characteristic numbers of M (i.e., the integrals of the characteristic classes) and the mixed equivariant characteristic classes of the fixed point data $(M^G, \mathcal{V})$.

For instance, by applying the localization formula to $u = 1$, we already obtain a non-trivial relation which is particularly simple when the fixed points are isolated:

$$\sum_{p \in M^G} \frac{1}{\prod \alpha_{i,p}} = 0.$$

In particular, if G is the circle group this turns into the relation

$$\sum_{p \in M^G} \frac{1}{\prod w_{i,p}} = 0 \tag{C.25}$$

on the integers $\omega_{i,p} \in \mathbb{Z}$, in the notation of Example C.55.

REMARK C.65. The relation (C.25) can be used to give a simple proof of a theorem of McDuff, [**McD2**], that a symplectic S^1-action on a compact symplectic four-manifold is Hamiltonian if and only if it has fixed points. (See [**Gin3**] for a detailed argument.)

REMARK C.66. There are other relations among the isotropy weights $\alpha_{i,p}$. For example, for an almost complex S^1-manifold with isolated fixed points,

$$\sum_{p \in M^G} \prod w_{i,p} = 0$$

as shown in [**Hat2**] by using the localization theorem in K-theory. It would be interesting to find explicitly all independent relations among weights $\alpha_{i,p}$ that hold for all (e.g., almost-complex) manifolds of a given dimension (cf. [**GZ2**]).

8.2. The equivariant Poincaré duality. Since the space $EG \times_G M$ is infinite dimensional, it is not clear how to extend to this space the Poincaré duality theorem relating homology and cohomology in complementary dimensions. There is however a different version of the Poincaré duality which does extend to equivariant cohomology H_G^* with real coefficients in a nearly trivial way: according to this version of the Poincaré duality (in its non-equivariant form), the standard pairing on real cohomology given by the integration over the manifold is non-degenerate. In a similar fashion, the equivariant cohomology $H_G^* \otimes \mathcal{Q}_G$ carries a non-degenerate $\mathcal{Q}_G$-bilinear paring. (Here, as above, $\mathcal{Q}_G$ is the field of fractions for $\mathcal{R}_G$.)

Let G be a torus acting on a compact oriented manifold M. Recall from Section 7 (see, in particular, (C.15) and (C.16)) that the integration over M, or equivalently the push-forward by the map $M \to \mathrm{pt}$, gives rise to a $\mathcal{Q}_G$-linear homomorphism $\int_M \colon H_G^*(M) \otimes \mathcal{Q}_G \to \mathcal{Q}_G$. For u and v in $H_G^*(M) \otimes \mathcal{Q}_G$, we set

$$\langle u, v \rangle_M = \int_M uv \in \mathcal{Q}_G.$$

This is a $\mathcal{Q}_G$-bilinear pairing on the vector space $H_G^*(M) \otimes \mathcal{Q}_G$ over $\mathcal{Q}_G$.

PROPOSITION C.67 (Equivariant Poincaré duality). *The pairing $\langle , \rangle_M$ is non-degenerate.*

Remark C.68. Recall that $H^*_G(M) \otimes \mathcal{Q}_G$ is a finite–dimensional vector space over $\mathcal{Q}_G$. Thus, the non-degeneracy condition is understood in the standard sense as for finite–dimensional vector spaces. More explicitly this means that for every $\mathcal{Q}_G$-bilinear map $\phi\colon H^*_G(M) \otimes \mathcal{Q}_G \to \mathcal{Q}_G$ there exists a unique $v \in H^*_G(M) \otimes \mathcal{Q}_G$ such that $\phi(x) = \langle v, x\rangle$ for all x. In particular, $u = 0$ if and only if $\langle u, x\rangle = 0$ for all x.

Proof of Proposition C.67. We have $H^*_G(M^G) \otimes \mathcal{Q}_G = H^*(M^G) \otimes \mathcal{Q}_G$, because the G-action on M^G is trivial. Hence, for M^G, the theorem follows from the ordinary Poincaré duality. We will derive the equivariant Poincaré duality for M from that for M^G.

Let

$$\phi\colon H^*_G(M) \otimes \mathcal{Q}_G \to \mathcal{Q}_G$$

be $\mathcal{Q}_G$-linear. Consider the $\mathcal{Q}_G$-linear map

$$\phi \circ j_*\colon H^*_G(M^G) \otimes \mathcal{Q}_G \to \mathcal{Q}_G,$$

where, as in the previous section,

$$j\colon M^G \to M$$

is the inclusion map.

Since the equivariant Poincaré duality holds for M^G, there exists a cohomology class $v \in H^*_G(M^G) \otimes \mathcal{Q}_G$ such that

$$\phi \circ j_*(x) = \langle v, x\rangle_{M^G}$$

for all $x \in H^*_G(M^G) \otimes \mathcal{Q}_G$. Then

$$\langle v, x\rangle_{M^G} = \left\langle j^*(j^*)^{-1}v, x\right\rangle_{M^G} = \left\langle (j^*)^{-1}v, j_*x\right\rangle_M .$$

(Note that j^* is an algebra isomorphism by the Borel localization theorem.) In other words, for $y = j_*x$,

$$\phi(y) = \langle u, y\rangle_M\,, \text{ where } u = (j^*)^{-1}v. \tag{C.26}$$

Since j_* is an isomorphism (over $\mathcal{Q}_G$), the identity (C.26) holds for *all* y. The uniqueness is a consequence of the fact that $H^*_G(M) \otimes \mathcal{Q}_G$ is finite–dimensional over $\mathcal{Q}_G$. ☐

The pairing $\langle,\rangle_M$ is, of course, also defined on $H^*_G(M)$ and is $\mathcal{R}_G$-bilinear on this space. This pairing ignores the torsion of $H^*_G(M)$.[2] It is not clear if an $\mathcal{R}_G$-linear map $\phi\colon H^*_G(M) \to \mathcal{R}_G$ can always be represented as the pairing with some $u \in H^*_G(M)$.

8.3. Generalizations to non-abelian groups. Let G be a compact Lie group which is not necessarily abelian. In this case, the fixed point set M^G can carry rather little information about $H^*_G(M)$, and Borel's localization theorem and its consequences, taken literally, do not hold for G-actions.

Example C.69. Let $M = G/\mathbb{T}$, where T is a maximal torus. Then $M^G = \emptyset$. However, $H^*_G(M) = H^*_{\mathbb{T}}(\mathrm{pt}) = H^*(B\mathbb{T}) = \mathcal{R}_{\mathbb{T}}$. This is a torsion–free module over $\mathcal{R}_G = \mathcal{R}_{\mathbb{T}}^W$, where W is the Weyl group. (In fact, $\mathcal{R}_{\mathbb{T}}$ is a free module over $\mathcal{R}_G$; see, e.g., [**Va**, Theorem 4.15.28].) Thus, in this example $H^*_G(M^G)$ gives no information on $H^*_G(M)$.

[2] We emphasize that here, as almost everywhere in this appendix, the torsion is understood over $\mathcal{R}_G$, not over $\mathbb{Z}$.

However, Borel's localization theorem and the Atiyah–Bott–Berline–Vergne localization extend to G-actions when M^G is replaced by a larger set. Let $M^{\max}$ be the set of points in M whose stabilizers have maximal rank:

$$M^{\max} = \{x \in M \mid \mathrm{rk}G_x = \mathrm{rk}G\}.$$

Clearly, $M^{\max}$ is invariant under the G-action. Note that $M^{\max}$ may fail to be a smooth submanifold of M.

THEOREM C.70. *Let G be a compact group acting on a compact manifold M. Borel's localization theorem (Theorem C.20) holds for M with M^G replaced by $M^{\max}$. If, in addition, $M^{\max}$ is smooth and M and $M^{\max}$ are orientable, the Poincaré duality (Proposition C.67) also holds for M and the Atiyah–Bott–Berline–Vergne localization formula* (C.17) *remains valid with M^G replaced by $M^{\max}$.*

REMARK C.71. The first assertion of this theorem is a very particular case of [**Hs**, Theorems III.1 and III.1′]. It is also worthwhile to notice that Theorem C.70 is *not* what is usually referred to as non-abelian localization in symplectic geometry.

PROOF OF THEOREM C.70. Let $\mathbb{T} \subset G$ be a maximal torus. Because $M^{\max}$ contains the fixed point set $M^{\mathbb{T}}$, the restriction of $H^*_{\mathbb{T}}(M)$ to $M^{\mathbb{T}}$ factors as the composition $H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}(M^{\max}) \to H^*_{\mathbb{T}}(M^{\mathbb{T}})$. It follows from Borel's localization (over $\mathbb{T}$) that $H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}(M^{\max})$ is an isomorphism modulo $\mathcal{R}_{\mathbb{T}}$-torsion.

By the first part of Theorem C.35, let us identify H^*_G with $(H^*_{\mathbb{T}})^W$ and $\mathcal{R}_G$ with $\mathcal{R}_{\mathbb{T}}^W$. Furthermore, observe that the $\mathcal{R}_G$-torsion submodule of $H^*_G(M)$ is exactly equal to the intersection of the $\mathcal{R}_{\mathbb{T}}$-torsion submodule of $H^*_{\mathbb{T}}(M)$ with $H^*_{\mathbb{T}}(M)^W$. It follows that

$$H^*_G(M) = H^*_{\mathbb{T}}(M)^W \to H^*_{\mathbb{T}}(M^{\max})^W = H^*_G(M^{\max})$$

is a monomorphism modulo $\mathcal{R}_G$-torsion.

Let us show that this map is in fact an epimorphism modulo $\mathcal{R}_G$-torsion. Assume that u is not in the image of this map. We need to show that $F \cdot u$ is in the image for some $F \in \mathcal{R}_G$. Since as we have pointed out above $H^*_{\mathbb{T}}(M) \to H^*_{\mathbb{T}}(M^{\max})$ is an isomorphism modulo torsion, $f \cdot u$ is in the image of $H^*_{\mathbb{T}}(M)$ for some $f \in \mathcal{R}_{\mathbb{T}}$. Because this image is an $\mathcal{R}_{\mathbb{T}}$-submodule, the element

$$F \cdot u, \quad \text{where} \quad F = \prod_{\gamma \in W} \gamma(f) \in \mathcal{R}_G,$$

is also in the image, i.e., $F \cdot u$ is the image of some $x \in H^*_{\mathbb{T}}(M)$. Note that $F \cdot u$ is W-invariant. Hence, $F \cdot u$ is the image of the cohomology class $\sum_{\gamma \in W} \gamma(x)/|W|$ which is W-invariant and hence an element of $H^*_G(M)$.

The identifications $H^*_G = (H^*_{\mathbb{T}})^W$ and, more generally, the W-action commute with push-forwards. Poincaré duality for G follows from Proposition C.67 and its proof. We leave the details to the reader as an exercise.

The Atiyah–Bott–Berline–Vergne localization formula is a formal consequence of the properties of push-forward, Borel's localization theorem, and the fact that the Euler class of the normal bundle to the fixed point set is invertible over $\mathcal{Q}_G$. Push–forwards for G-actions have the same properties as for torus actions. Borel's localization theorem has just been extended to G-actions with $M^{\max}$ taken in place of M^G.

It remains to prove that $e^G(\mathcal{V}_{M^{\max}})$ is invertible in $H^*_G(M)\otimes\mathcal{R}_G$. Let F be a connected component of $M^{\max}$. Then, over the field of rational functions,

$$e^G(\mathcal{V}_F)=e^{\mathbb{T}}(\mathcal{V}_F)=e^{\mathbb{T}}(\mathcal{V}_F)|_{F^{\mathbb{T}}} \tag{C.27}$$

under the sequence of identifications

$$H^*_G(F)\otimes\mathcal{Q}_G\xrightarrow{\cong}(H^*_{\mathbb{T}}(F)\otimes\mathcal{Q}_{\mathbb{T}})^W\xrightarrow{\cong}(H^*_{\mathbb{T}}(F^{\mathbb{T}})\otimes\mathcal{Q}_{\mathbb{T}})^W.$$

(Note that W can act non-trivially on both terms in $H^*_{\mathbb{T}}(F^{\mathbb{T}})\otimes\mathcal{Q}_{\mathbb{T}}$.) Furthermore, $e^{\mathbb{T}}(\mathcal{V}_F)|_{F^{\mathbb{T}}}$ is still given by (C.13), where the product is over all $\mathbb{T}$-irreducible components of $\mathcal{V}_F|_{F^{\mathbb{T}}}$. Since none of these components is a trivial representation, the leading term is non-zero and $e^{\mathbb{T}}(\mathcal{V}_F)|_{F^{\mathbb{T}}}$ is invertible as a rational function. By (C.27), the class $e^G(\mathcal{V}_F)$ is also invertible.

This completes the proof of the Theorem C.70. □

Remark C.72. It would be interesting to see if, or under what conditions, $H^*_G(M^{\max})$ is a torsion–free $\mathcal{R}_G$-module.

9. Equivariant homology

In this section we give a geometrical interpretation of the *equivariant homology* $H^G_*(M)=H_*(EG\times_G M)$. Although the main result of this section holds for homology with any coefficients, for the sake of simplicity we restrict our attention to the coefficient group $\mathbb{Z}$.

Recall that the homology of a CW-complex X can be defined by the following construction. Consider oriented manifolds Σ with singularities in codimension two or greater. (See [**RS**] and references therein for piecewise-linear versions of these definitions.) We assume Σ to be compact but possibly with boundary $\partial\Sigma$. Note that $\partial\Sigma$ is again an oriented manifold with singularities in codimension two or greater. Let $C_k(X)$ be a free group with generators $f\colon\Sigma\to X$ for all Σ and f, with the standard convention about the orientations. It is clear that $C_*(X)$ is a complex with differential $\partial f=f|_{\partial\Sigma}$. The homology of this complex is equal to the ordinary singular homology $H_*(X;\mathbb{Z})$.

Now let M be a CW-complex with an action of a compact group G. Consider the complex $C^G_*(M)$ defined as above, but now with Σ carrying a *free* G-action and f being equivariant.

Remark C.73. Similar maps of principal G-bundles into a symplectic G-manifold have been used to define pseudo-holomorphic curves and Gromov–Witten invariants for symplectic quotients. See [**GaSa, Mu1, CGS**].

Theorem C.74. *The homology of the complex $C^G_*(M)$ in degree q is naturally isomorphic to the equivariant homology $H^G_{q-n}(M;\mathbb{Z})$, where $n=\dim G$.*

Proof. Consider the homomorphism of complexes

$$\Phi\colon C^G_*(EG\times M)\to C_{*-n}(EG\times_G M)$$

arising from taking the quotient by G. Namely, let Σ be equipped with a free G-action and $F\colon\Sigma\to EG\times M$ be G-equivariant. Then F descends to the map $f\colon\Sigma/G\to EG\times_G M$, where Σ/G is again a manifold with singularities in codimension two or greater. By setting $\Phi(F)=f$ we obtain a homomorphism of complexes.

The homomorphism Φ is in fact an isomorphism. To define its inverse, denote by ρ the principal G-bundle $EG\times M\to EG\times_G M$. Let $f\colon\Sigma\to EG\times_G M$

be an element of $C_{*-n}(EG \times_G M)$ and let $\tilde{\Sigma}$ be the total space of the pull-back G-principal bundle $f^*\rho\colon \tilde{\Sigma} \to \Sigma$. It is clear that f lifts to a G-equivariant map $F\colon \tilde{\Sigma} \to EG \times M$, where $\tilde{\Sigma}$ is a manifold with singularities in codimension two or greater. By the construction, $\Phi(F) = f$.

As a consequence, the homology of $C_*^G(EG \times M)$ is naturally isomorphic to $H_{*-n}^G(M; \mathbb{Z})$.

It remains to show that $C_*^G(EG \times M)$ has the same homology as $C_*^G(M)$. Every map $F \in C_*^G(EG \times M)$ has the form (κ, ϕ) where $\phi \in C_*^G(M)$ and $\kappa\colon \Sigma \to EG$ is the lift of a classifying map $\Sigma/G \to BG$. Since the G-action on Σ determines the classifying map up to a homotopy, it also determines κ up to a G-equivariant homotopy. This shows that the projection $F \mapsto \phi$ induces an isomorphism in the homology of these complexes. (We leave the details to the reader as an exercise.) $\square$

REMARK C.75. A similar construction applies to maps of genuine manifolds (without singularities) and gives a geometrical interpretation of *equivariant bordisms* of M. Namely, the ring of equivariant bordisms of M, understood as ordinary bordisms of $EG \times_G M$, can be described via equivariant mappings to M of manifolds with free G-actions.

REMARK C.76. It would be interesting to understand what "homology theory" results from the above construction when free actions are replaced by locally free actions. (For example, a regular level of a moment map for a torus action gives an element of such a homology theory.)

REMARK C.77. For every n, there exists an obvious non-degenerate pairing $H_G^n(M; \mathbb{R}) \times H_n^G(M; \mathbb{R}) \to \mathbb{R}$. However, it is not clear how $H_*^G(M)$ fits into the picture of equivariant Poincaré duality in the context of Section 8.2.