APPENDIX A

Signs and normalization conventions

In this appendix we describe our conventions regarding signs and factors of 2π. We discuss the representation of G on $C^\infty(M)$ when M is a G-manifold, the integral weight lattice $\mathbb{Z}_G^*$ for a torus G, connections and curvature, and integral equivariant cohomology.

1. The representation of G on $\mathbf{C}^\infty(\mathbf{M})$

Consider an action τ of a group G on a manifold M. Thus for each $g \in G$ we have a diffeomorphism $\tau_g \colon M \to M$, and $\tau_{gh} = \tau_g \tau_h$. The action τ gives rise to a representation of G on $C^\infty(M)$ according to the rule

$$T \colon g \mapsto \left(\tau_g^{-1}\right)^* \tag{A.1}$$

or, more explicitly,

$$\left(T(g)f\right)(x) = f\left(\tau_g^{-1}x\right)$$

for $f \in C^\infty(M)$ and $x \in M$. This assignment ensures that $T(gh) = T(g)T(h)$ so that T is indeed a representation.

If we attempt, instead, to make the simpler definition where $g \in G$ acts on $C^\infty(M)$ by the pull-back map $\tau_g^* \colon C^\infty(M) \to C^\infty(M)$, we would get an *anti*-representation of G on $C^\infty(M)$, for $(\tau_g\tau_h)^* = \tau_h^*\tau_g^*$. At the infinitesimal level, this becomes an issue of signs. Namely, to each element ξ of $\mathfrak{g}$ one associates a one-parameter group $t \mapsto \tau_{(\exp t\xi)}$ of diffeomorphisms of M whose infinitesimal generator is denoted by ξ_M. At the infinitesimal level, the fact that $g \mapsto \tau_g^*$ is an anti-representation is reflected by the identity

$$[\xi_M, \eta_M] = -[\xi, \eta]_M, \tag{A.2}$$

which says that the map

$$\mathfrak{g} \to \mathrm{Vect}(M), \qquad \xi \mapsto \xi_M,$$

is an *anti*-isomorphism of Lie algebras. If the representation of G on $C^\infty(M)$ is defined correctly, as in (A.1), the corresponding infinitesimal representation becomes

$$\xi \mapsto -L_{\xi_M},$$

which is a homomorphism of Lie algebras.

In this monograph G will be, for the most part, an *abelian* Lie group. As a consequence, the sign problem, formally speaking, does not arise here, for both the right- and left-hand sides of (A.2) are zero. However, if one wants to be careful about these distinctions, one has to be careful about sign conventions in simpler situations where these distinctions are frequently ignored. For instance, an example which we often encounter is a linear representation of the torus on $\mathbb{C}^d$ and the resulting representation of G on $C^\infty(\mathbb{C}^d)$:

EXAMPLE A.1. Let G be a torus, and let τ be the representation of G on $\mathbb{C}^d$ defined by

$$\tau_{(\exp\xi)} z = \left(e^{-i\alpha_1(\xi)} z_1, \ldots, e^{-i\alpha_d(\xi)} z_d\right), \tag{A.3}$$

the α_i's being weights of G (see Section 2 below). From τ we get a representation of G on $C^\infty(\mathbb{C}^d)$, and if we define this representation *correctly* (as in (A.1)), the coordinate functions $z_1, \ldots, z_d$ transform by the formula

$$z_k \mapsto e^{i\alpha_k(\xi)} z_k, \tag{A.4}$$

i.e., with weights having *opposite signs* to the weights of the representation (A.3). Thus, at first glance, they will look as if they are transforming according to the wrong sign! We will have to exercise great care not to confuse the action of G on $\mathbb{C}^d$ defined by (A.3) with the representation of G on $C^\infty(\mathbb{C}^d)$ defined by (A.4).

For further discussion of linear representations of G on $\mathbb{C}^d$, see Section 4 of Chapter 4.

2. The integral weight lattice

Another problem which we often face concerns factors of 2π. Throughout most of this book G is a torus, i.e., a Lie group isomorphic to $(S^1)^k$ for some k. Hence the exponential map, $\exp\colon \mathfrak{g} \to G$, is a group epimorphism. The *group lattice* of G is the lattice $\mathbb{Z}_G = \ker \exp$ in $\mathfrak{g}$. The *weight lattice* $\mathbb{Z}_G^*$ parametrizes the characters of G, i.e., the homomorphisms $G \to S^1$. Such a homomorphism is determined by its differential at the identity element $e \in G$. This differential is a linear map from the tangent space $T_eG = \mathfrak{g}$ to the tangent space T_1S^1, hence it is an element of $\mathfrak{g}^* \otimes \mathrm{Lie}(S^1)$. To realize $\mathbb{Z}_G^*$ as a lattice in the dual space $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, \mathbb{R})$, we need to make a single choice: we need to choose an isomorphism

$$\mathrm{Lie}(S^1) \cong \mathbb{R}. \tag{A.5}$$

Equivalently, we need to specify an exponential map

$$\exp\colon \mathbb{R} \to S^1. \tag{A.6}$$

The weight lattice $\mathbb{Z}_G^*$ then becomes a lattice in $\mathfrak{g}^*$ which is dual to the group lattice $\mathbb{Z}_G \subset \mathfrak{g}$ in the sense that

$$\mathbb{Z}_G^* = \{\alpha \in \mathfrak{g}^* \mid \langle \alpha, \xi \rangle \in \mathbb{Z}_{S^1} \text{ for all } \xi \in \mathbb{Z}_G\},$$

where $\mathbb{Z}_{S^1} \subset \mathbb{R}$ is the kernel of the exponential map (A.6).

We choose the isomorphism (A.5) to be

$$\mathrm{Lie}(S^1) = \mathbb{R} \quad \text{via} \quad \frac{\partial}{\partial\theta} \longleftrightarrow 1, \tag{A.7}$$

where $\theta \mod 2\pi$ is a coordinate on S^1, so that the exponential map (A.6) becomes

$$\exp\colon \theta \mapsto e^{i\theta}.$$

We then have

$$\mathbb{Z}_{S^1} = 2\pi\mathbb{Z},$$

and

$$\mathbb{Z}_G^* = \{\alpha \in \mathfrak{g}^* \mid \langle \alpha, \xi \rangle \in 2\pi\mathbb{Z} \text{ for all } \xi \in \mathbb{Z}_G\}.$$

A weight $\alpha \in \mathbb{Z}_G^*$ determines a homomorphism $G \to S^1$ by $\exp(\xi) \mapsto e^{i\alpha(\xi)}$. (Note that, if $\xi \in \ker \exp$, then $\alpha(\xi) \in 2\pi\mathbb{Z}$.)

In particular, for $G = S^1$, we have $\mathfrak{g} = \mathfrak{g}^* = \mathbb{R}$, the group lattice is $\mathbb{Z}_G = 2\pi\mathbb{Z}$, and the weight lattice is $\mathbb{Z}_G^* = \mathbb{Z}$. A moment map for a circle action τ becomes a real valued function $\Phi\colon M \to \mathbb{R}$ satisfying $d\Phi = \iota(\xi_M)\omega$, where ξ_M is the vector field $\frac{d}{d\theta}\tau(e^{i\theta})$. The weight $m \in \mathbb{Z}$ parametrizes the homomorphism $\lambda \mapsto \lambda^m$ of S^1.

Our normalization convention simplifies many formulas by eliminating factors of 2π. Another advantage of this convention is that it leads to a notion of curvature which becomes the standard Riemannian geometry curvature when applied to the unit circle bundle of a two-dimensional Riemannian manifold. See the next section for details.

Remark A.2. Another possible convention would have been to choose the exponential map (A.6) to be $x \mapsto e^{2\pi\sqrt{-1}x}$. The lattice $\mathbb{Z}_G^*$ would have become the dual lattice to $\mathbb{Z}_G \subset \mathfrak{g}$ in the standard sense: pairings between elements of $\mathbb{Z}_G$ and $\mathbb{Z}_G^*$ would have been integers. The curvature of a circle bundle would have becomes a real two-form which represents the first Chern class of the bundle.

3. Connection and curvature for principal torus bundles

Let $\pi\colon P \to M$ be a principal K-bundle, where K is a torus with Lie algebra $\mathfrak{k}$. (More generally, we may allow P to be an orbi-bundle, meaning that K acts on P locally freely and M is the orbifold P/K. See Corollary B.31.) Let ζ_P, for $\zeta \in \mathfrak{k}$, be the vector fields on P that generate the principal action. A *connection form* is a $\mathfrak{k}$-valued one-form Θ on the total space P which is K-invariant and whose restriction to the orbits is the tautological (Maurer-Cartan) form

$$\Theta(\zeta_P) = \zeta \quad \text{for all } \zeta \in \mathfrak{k}. \tag{A.8}$$

A connection form always exists; see Corollary B.37.

Lemma A.3. *There exists a two-form F on M such that*

$$\pi^* F = -d\Theta. \tag{A.9}$$

Proof. We need to show that the two-form $d\Theta$ is basic, i.e., is K-invariant and is horizontal.

It is invariant because Θ is invariant.

It is horizontal because

$$\iota(\zeta_P)d\Theta = -d\iota(\zeta_P)\Theta = 0.$$

Here, the first equality uses Cartan's formula $L_{\zeta_P} = d\iota(\zeta_P) + \iota(\zeta_P)d$ and the fact that Θ is invariant, and the second equality uses the fact that $\iota(\zeta_P)\Theta \equiv \zeta$ is constant. □

Remark A.4. Here is a fancier way to show that the exterior derivative $d\Theta$ is basic. The function $\Phi^\zeta = \Theta(\zeta_P) \equiv \zeta$ is a moment map for $-d\Theta$ (see Example 2.8). Because $d\Theta$ is a $\mathfrak{k}$-valued two-form, this moment map takes values in $\mathfrak{k}^* \otimes \mathfrak{k}$, and we have just shown that it takes the constant value *identity* $\in \mathfrak{k}^* \otimes \mathfrak{k} \cong \mathrm{Hom}(\mathfrak{k}, \mathfrak{k})$. By pre-symplectic reduction (Theorem 5.1), $d\Theta$ descends to a ($\mathfrak{k}$-valued) closed two-form on the quotient.

Definition A.5. The *curvature* of the connection Θ is the $\mathfrak{k}$-valued two-form F on M satisfying (A.9).

Let us now specialize to $K = S^1$ and identify its Lie algebra $\mathfrak{k}$ with $\mathbb{R}$ using (A.7). Then a connection one-form is a real-valued one-form Θ such that $\Theta(\partial/\partial\theta) = 1$, where $\partial/\partial\theta$ generates the principal circle action, and the curvature is a real-valued two-form. If P is the unit circle bundle of a two-dimensional Riemannian manifold and Θ is the Levi-Civita connection, then our convention gives the same curvature as in the usual Riemannian geometry theory. For example, for $M = S^2$ with its standard metric, the unit circle bundle has curvature equal to the standard area form. This justifies our sign and normalization conventions.

Remark A.6. Many authors define curvature by $\pi^* F = d\Theta$. With this sign convention (which is opposite from ours), the curvature gives the *negative* of "infinitesimal holonomy" (see below). Also, it is common to consider these Θ and F as taking values in $\mathrm{Lie}(S^1) = i\mathbb{R}$ and then to pass to the real-valued two-form $\omega = \frac{i}{2\pi}F$. This two-form is the same as we get would from our definition of curvature, but with the convention of Remark A.2. For the standard Riemannian two-sphere, this two-form ω is equal to $1/2\pi$ times the standard area form.

Finally, let us recall how to interpret the curvature F as *infinitesimal holonomy.*

For any two horizontal vector fields $\tilde{u}$ and $\tilde{v}$,

$$-d\Theta(\tilde{u},\tilde{v}) = -\left(\tilde{u}\Theta(\tilde{v}) - \tilde{v}\Theta(\tilde{u}) - \Theta([\tilde{u},\tilde{v}])\right) = \Theta([\tilde{u},\tilde{v}]), \tag{A.10}$$

where the first equality holds for *all* one-forms and vector fields, and the second equality follows from the horizontality of $\tilde{u}$ and $\tilde{v}$, which exactly means that $\Theta(\tilde{u}) \equiv \Theta(\tilde{v}) \equiv 0$. Let X_t and Y_t denote the flows on P generated by $\tilde{u}$ and $\tilde{v}$.

Lemma A.7. *The one-parameter family of maps of P given by the diffeomorphisms $Y_{-\sqrt{t}}X_{-\sqrt{t}}Y_{\sqrt{t}}X_{\sqrt{t}}$, for $t \in \mathbb{R}$, is differentiable at $t = 0$, and*

$$\left.\frac{d}{dt}\right|_{t=0} Y_{-\sqrt{t}}\,X_{-\sqrt{t}}\,Y_{\sqrt{t}}\,X_{\sqrt{t}} = [\tilde{u},\tilde{v}]. \tag{A.11}$$

Now, let u and v be vector fields with constant coefficients with respect to some local coordinate system. Let $\tilde{u}$ and $\tilde{v}$ be their horizontal lifts to P. The holonomy over the parallelogram with sides $\sqrt{t}u$, $\sqrt{t}v$, $-\sqrt{t}u$, and $-\sqrt{t}v$ (taken in this order) exactly sends p to $Y_{-\sqrt{t}}\,X_{-\sqrt{t}}\,Y_{\sqrt{t}}\,X_{\sqrt{t}}\,p$. By (A.11), (A.10), and (A.9), we can interpret $F(u,v)$ as the "holonomy along the infinitesimal parallelogram generated by u and v".

Proof of Lemma A.7. Restrict attention to a coordinate neighborhood of a point $p \in P$. The definition of the Lie bracket of vector fields gives

$$[\tilde{u},\tilde{v}](p) = \left.\frac{d^2}{dt\,ds}\right|_{t=s=0} (Y_s X_t p - X_s Y_t p). \tag{A.12}$$

This is equal to (A.11), because they are both equal to

$$\langle \tilde{v}'(p), \tilde{u}(p)\rangle - \langle \tilde{u}'(p), \tilde{v}(p)\rangle\,, \tag{A.13}$$

where $\tilde{u}'$ and $\tilde{v}'$ are the differentials of $\tilde{u}$ and $\tilde{v}$ and

$$\langle\,,\,\rangle :\ \left(T_p^*P \otimes T_pP\right) \times T_pP \to T_pP$$

is the natural pairing. One gets (A.13) by substituting in (A.11) and (A.12) the Taylor expansions

$$X_t(q) = q + t\tilde{u}(q) + \frac{1}{2}t^2\,\langle \tilde{u}'(q), \tilde{u}(q)\rangle + \mathcal{O}(t^3)$$

and

$$\tilde{u}(q(t)) = \tilde{u}(q(0)) + t\,\langle \tilde{u}'(q(0)), q'(0)\rangle + \mathcal{O}(t^2)$$

as well as those for $Y_t(q)$ and $\tilde{v}(q(t))$ as many times as needed. $\square$

4. Curvature and Chern classes

Let $\pi\colon P \to M$ be a principal K-bundle, for K a torus. Let F be its curvature with respect to some connection one-form Θ.

LEMMA A.8. *The cohomology class of the curvature F is independent of the connection one-form Θ.*

PROOF. Let Θ and Θ' be any two connection one-forms. The definition of a connection form implies that the difference $\Theta' - \Theta$ is basic, so there exists a $\mathfrak{k}$-valued one-form α on M such that $\Theta' - \Theta = \pi^*\alpha$. Then $\pi^*(F' - F) = d\Theta - d\Theta' = -\pi^* d\alpha$, so $F' = F - d\alpha$ is in the same de Rham cohomology class as F. $\square$

DEFINITION A.9. The *curvature class* of $\pi\colon P \to M$ is the de Rham cohomology class $[F]$.

LEMMA A.10. *The curvature class $[F]$ belongs to the image of the natural homomorphism*

$$H^2(M; \mathbb{Z}_K) \to H^2(M; \mathfrak{k}). \qquad \text{(A.14)}$$

PROOF. The short exact sequence $0 \to \mathbb{Z}_K \to \mathfrak{k} \to K \to 0$ gives a long exact sequence of Čech cohomology groups, from which we get an isomorphism $\check{H}^1(M;K) \cong \check{H}^2(M;\mathbb{Z}_K)$. The transition maps of the principal bundle $P \to M$ represent an element of $\check{H}^1(M;K)$. Let $\check{c}$ denote the corresponding element of $\check{H}^2(M;\mathbb{Z}_K)$. Then $\check{c}$ maps to $[F]$ under (A.14). (See [**BT1**].) $\square$

When $K = S^1$, so that $\mathfrak{k} = \mathbb{R}$ and $\mathbb{Z}_K = 2\pi\mathbb{Z}$, the *Chern class of the circle bundle* $P \to M$ is $\frac{1}{2\pi}\check{c} \in H^2(M;\mathbb{Z})$. Its image under the natural homomorphism

$$i\colon H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R}) \qquad \text{(A.15)}$$

is $1/2\pi$ times the curvature class.

The *Chern class of a complex line bundle* is defined as the Chern class of its unit circle bundle with respect to any fiberwise Hermitian metric. In particular, we may consider the associated line bundle

$$\mathbb{L} = P \times_K \mathbb{C}_\alpha$$

where P is a principal K (orbi-)bundle and $\mathbb{C}_\alpha$ denotes the K-action on $\mathbb{C}$ by weight $\alpha \in \mathbb{Z}_K^*$.

LEMMA A.11. *The curvature class of the principal bundle, $c(P) \in H^2(M;\mathfrak{k})$, and the Chern class of the associated line bundle, $c_1(\mathbb{L}) \in H^2(M;\mathbb{Z})$, are related by*

$$i(c_1(\mathbb{L})) = \frac{1}{2\pi}\langle \alpha, c(P)\rangle\,.$$

PROOF. The unit circle bundle in $\mathbb{L}$ is $P \times_K S^1$ where K acts on S^1 with weight α. A ($\mathfrak{k}$-valued) connection one-form Θ on P induces a (real valued) connection one-form on this circle bundle given by

$$\langle \alpha, \Theta\rangle + d\theta,$$

where $\theta \mod 2\pi$ is a coordinate on S^1. The lemma then follows from the definitions of curvature and Chern classes. $\square$

5. Equivariant curvature; integral equivariant cohomology

Let a Lie group G act on a principal U(1)-bundle $\pi\colon P \to M$ by bundle automorphisms, generated by vector fields

$$\xi_P, \qquad \xi \in \mathfrak{g}.$$

Let Θ be a G-invariant connection one-form on P, considered as a real-valued one-form as explained earlier. Such a connection always exists if G acts properly; see Corollary B.38. Its *equivariant curvature* is the formal difference

$$\omega - \Phi,$$

where ω is a two-form on M such that $\pi^*\omega = -d\Theta$ and where $\Phi\colon M \to \mathfrak{g}^*$ is such that $\pi^*\Phi^\xi = \Theta(\xi_P)$ for all $\xi \in \mathfrak{g}$.

The *equivariant curvature class* of P is defined to be the equivariant cohomology class $c_G(P) = [\omega - \Phi] \in H^*_G(M)$. It is independent of the choice of connection. We leave the details to the reader, and refer to Definition 2.15 and Appendix C for the relevant definitions.

We have the following notion of integrality in equivariant cohomology in the Cartan model. When G is a torus, $H^*_G(\text{point};\mathbb{R}) = H^*(BG;\mathbb{R})$ are the polynomial functions on $\mathfrak{g}$, and the image of $H^*_G(\text{point};\mathbb{Z}) \to H^*_G(\text{point};\mathbb{R})$ consists of the polynomial functions on $\mathfrak{g}$ whose values on the group lattice $\mathbb{Z}_G$ are integers. If $[\omega - \Phi]$ is the equivariant curvature class of a G-equivariant circle bundle $P \to M$, then $\frac{1}{2\pi}[\omega - \Phi]$ is integral.