APPENDIX I

The Kawasaki Riemann–Roch formula

The Kawasaki Riemann–Roch theorem is a version of the Riemann–Roch theorem which applies to orbifolds. The goal of this appendix is to explain the formulation of this theorem.

1. Todd classes

Throughout this appendix we will make frequent use of the "splitting principle". For a precise formulation and justification of this principle we refer to [**BT1**, Section 21] or Section 5 of Appendix C. We refer to Appendix C for an introduction to equivariant cohomology and equivariant characteristic classes. A somewhat free translation of the splitting principle is as follows.

Splitting Principle: *If M is a compact manifold and $E \to M$ a complex vector bundle one can "in making computations" assume that E splits into a direct sum of complex line bundles*

$$E = \mathbb{L}_1 \oplus \ldots \oplus \mathbb{L}_r.$$

We will also need the *equivariant* version of the splitting principle: Let G be an n-dimensional torus, M a G-manifold M, and $E \to M$ a G-equivariant vector bundle. (This means that E is equipped with a lift of the G-action on M.) Then one can assume "in making computations" that the above splitting is G-equivariant.

This principle greatly facilitates the computation of the basic topological invariants of E. For instance, the *total Chern class* $c(E) = 1 + c_1(E) + \ldots + c_r(E)$ of E (where $r = \dim_{\mathbb{C}} M$) can be found by the product formula:

$$c(E) = \prod_{k=1}^{r} \left(1 + c_1(\mathbb{L}_k)\right). \tag{I.1}$$

In other words, $c_m(E)$ is the elementary symmetric polynomial of degree m in $c_1(\mathbb{L}_1), \ldots, c_1(\mathbb{L}_r)$. Note also that $1 + c_1(\mathbb{L}_k)$ is the total Chern class of $\mathbb{L}_k$ and hence this formula can be read as $c(E) = \prod c(\mathbb{L}_k)$.

A slightly more complicated invariant is the *Todd class* of E which is defined as follows. Let

$$\tau(x) = \frac{x}{1 - e^{-x}}$$

be the "classical" Todd function. Recall that its Taylor series expansion about the origin is

$$\tau(x) = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k},$$

where B_k is the kth Bernoulli number. Since $c_1(\mathbb{L}_i)^k = 0$ for $k > \frac{1}{2} \dim M$, the substitution $x = c_1(\mathbb{L}_i)$ converts this series into a finite sum, and we define the

Todd class of E to be

$$\mathrm{Td}\,(E) = \prod_{i=1}^{r} \tau\,(c_1(\mathbb{L}_i))\,. \tag{I.2}$$

Notice also that the right-hand side can be written as a polynomial in the elementary symmetric functions of the first Chern classes $c_1(\mathbb{L}_i)$ and hence, by the product formula, as a polynomial in the Chern classes $c_k(E)$.

One useful consequence of the splitting principle is the following multiplicative property of the Todd class.

Theorem I.1. *Let E' and E'' be vector bundles over M. Then*

$$\mathrm{Td}\,(E' \oplus E'') = \mathrm{Td}\,(E')\,\mathrm{Td}\,(E''). \tag{I.3}$$

Proof. Let $E = E' \oplus E''$. From splittings $E' = \oplus \mathbb{L}'_i$ and $E'' = \oplus \mathbb{L}''_i$ we obtain a splitting of E into the sum of the line bundles $\mathbb{L}'_i$ and $\mathbb{L}''_i$. Hence

$$\begin{aligned} \mathrm{Td}\,(E) &= \left(\prod \tau\,(c(\mathbb{L}'_i))\right)\left(\prod \tau\,(c(\mathbb{L}''_i))\right) \\ &= \mathrm{Td}\,(E')\,\mathrm{Td}\,(E''). \end{aligned}$$

□

In particular, Theorem I.1 implies that the Todd class is stable:

$$\mathrm{Td}\,(E \oplus \mathbb{C}^k) = \mathrm{Td}\,(E).$$

Now let M be a compact pre-quantizable manifold with a pre-quantization $(\mathbb{L}, \langle , \rangle, \nabla)$ and a stable complex structure J. (See Section 2 of Chapter 6.)

In Section 7 of Chapter 6 we defined the stable complex quantization $\mathcal{Q}(M)$ of M as the index of an associated Dirac operator D. That is, $\mathcal{Q}(M)$ is a finite–dimensional virtual vector space equal to $\ker D - \operatorname{coker} D$. However, since the Grothendieck group of virtual vector spaces is canonically isomorphic to $\mathbb{Z}$ via the isomorphism dim, we can think of $\mathcal{Q}(M)$ as the integer $\dim \mathcal{Q}(M)$, which is, by definition, equal to the index of D.

The Atiyah–Singer index theorem applied to this operator implies that

$$\mathcal{Q}(M) = \int e^{c_1(\mathbb{L})}\,\mathrm{Td}\,(TM). \tag{I.4}$$

Remark I.2. Probably the simplest of the many proofs of this is the K-theoretic proof in [**ASe**]. See also [**BGV**, Chapter 4] and [**Du**].

2. The Equivariant Riemann–Roch Theorem

In this section we discuss the *equivariant* version of the Atiyah–Singer formula (I.4). Let us begin by recalling how the *equivariant* Chern class of a line bundle is defined. (See also Section 7 of Appendix C for a different construction.)

2.1. Equivariant Todd classes. Let G be an n-torus, M a G-manifold, and $\pi\colon P \to M$ a principal S^1-bundle on which G acts by bundle automorphisms. Pick a G-invariant connection form $\Theta \in \Omega^1(P)^G$ on P (which we assume to be real-valued). Then the first equivariant Chern class of P, with real coefficients, is the equivariant cohomology class $\frac{1}{2\pi}[\omega^G]$, where ω^G is the equivariant curvature, which is defined by $d_G\Theta = -\pi^*\omega^G$. We denote this class by $c_1^G(\mathbb{L}) \in H^2_G(M)$, where $\mathbb{L} = P \times_{S^1} \mathbb{C}$ is the complex line bundle associated with P. (See Section 2 of

Chapter 6 and Appendix A.) It is easy to see that although ω^G depends on the choice of Θ, the class $[\omega^G]$ is well defined.

If G acts *trivially* on M, this class is particularly easy to describe. (See also Example C.41.) In this case the group G acts on $\mathbb{L}_p$ for all $p \in M$, and (if M is connected) this action is independent of p. Denote by α the weight of this action. Recall that, since the G-action on M is trivial, $H_G^*(M)$ can be thought of as the space of polynomials on $\mathfrak{g}$ with values in $H^*(M)$. (See Example C.6.) In particular, $H_G^2(M) = \mathfrak{g}^* \oplus H^2(M)$.

LEMMA I.3. *For the trivial G-action, $c_1^G(\mathbb{L}) = c_1(\mathbb{L}) - \frac{1}{2\pi}\alpha$, where $c_1(\mathbb{L})$ is the ordinary first Chern class of $\mathbb{L}$.*

PROOF. The action of $g = \exp(t\xi)$ on P is just multiplication by $e^{\sqrt{-1}t\alpha(\xi)}$. Hence

$$\xi_P = \alpha(\xi)\frac{\partial}{\partial\theta},$$

where $\partial/\partial\theta$ is the generator of the action of S^1 on P. Thus

$$d_G\Theta(\xi_P) = \iota(\xi_P)\Theta + d\Theta = \alpha(\xi) - \pi^*\omega,$$

where Θ is a connection form and ω is the curvature for Θ. Because $\frac{1}{2\pi}[\omega] = c_1(\mathbb{L})$, the lemma follows. □

Next, let E be a complex G-equivariant vector bundle over a G-manifold M. By the splitting principle (see Section 1 of this appendix and Section 5 of Appendix C), we can assume that there is an equivariant splitting

$$E = \mathbb{L}_1 \oplus \ldots \oplus \mathbb{L}_r.$$

Thus we define the equivariant Chern classes $c_k^G(E)$ of E, as before, to be the symmetric polynomials in $c_1^G(\mathbb{L}_i)$, $i = 1, \ldots, r$. Equivalently, it may be defined by (I.1) with all Chern classes being now equivariant.

To define the equivariant Todd class of $\mathbb{L}_i$, we substitute $c_1^G(\mathbb{L}_i)$ into the infinite series for $\tau(x)$ and, similarly to (I.2), use the splitting principle (and the product formula) to define the equivariant Todd class of E. Thus,

$$\operatorname{Td}^G(E) = \prod_{i=1}^{r} \tau(c_1^G(\mathbb{L}_i)). \tag{I.5}$$

This definition deserves some discussion. As we have pointed out above, the usual Todd class of E is a *polynomial* in the Chern classes. This is no longer the case for the equivariant Todd class. Since the high powers of $c_1^G(\mathbb{L}_i)$ are no longer zero in general, $\operatorname{Td}(E)$ is an infinite series, not just a polynomial, in $c_k^G(E)$ or in $c_1^G(\mathbb{L}_i)$. Thus, $\operatorname{Td}^G(E)$ should be considered as an element of the equivariant cohomology of M over the ring of infinite power series on $\mathfrak{g}$; see Remark C.14. Furthermore, taking the representatives for $c_1(\mathbb{L}_i)$ described above, we obtain a differential form representing $\operatorname{Td}^G(E)$ which is a power series on $\mathfrak{g}$ convergent for small values of $\xi \in \mathfrak{g}$. Hence, $\operatorname{Td}^G(E)$ can also be thought of as an element of the equivariant cohomology of M over the ring of (the germs of) analytic functions on $\mathfrak{g}$. This is the point of view which we will adopt in what follows.

2.2. The equivariant index theorem. Suppose now that M is a G-manifold equipped with pre-quantization data $(\mathbb{L}, \langle , \rangle , \nabla)$ and with a G-equivariant stable complex structure. From this data we obtain a virtual representation $\mathcal{Q}(M)$ of G, as described in Section 7 of Chapter 6 and Section 3.5 of Appendix D. Denote by χ the character of this representation. Recall that χ is a smooth function (in fact a trigonometric polynomial) on G. The composition $\chi \circ \exp$ is then a smooth function on $\mathfrak{g}$.

THEOREM I.4 (The equivariant index theorem). *On a neighborhood of* $0 \in \mathfrak{g}$*,*

$$\chi \circ \exp = \int_M e^{c_1^G(\mathbb{L})} \operatorname{Td}^G(TM). \tag{I.6}$$

This version of the equivariant index theorem is due to Bismut (for the standard Dirac operator); see, e.g., [**Bi**] and also [**BGV**, Chapter 8] and references therein. An adaptation of his proof to the Dolbeault-Dirac operator case can be found in [**Du**].

REMARK I.5. Let us compare formula (I.6) with the topological Riemann–Roch formula (6.42) which has the same right-hand side as (I.6). The topological Riemann–Roch formula (6.42) establishes an equality of two topological objects, namely, that the right-hand side of (I.6) is equal to $\operatorname{ch}^G(p_!([\mathbb{L}]))$. This is equivalent to that the Taylor expansion of the character of $p_!([\mathbb{L}])$ is given by the integral on the right-hand side of either of these formulas. On the other hand, the index theorem states that the virtual representation obtained as the equivariant index of the Dirac operator has the character given by this integral. Matching these formulas, we see that $\mathcal{Q}(M) = p_!([\mathbb{L}])$, which was the main motivation for the topological definition of the quantization as the push–forward in Section 8 of Chapter 6.

The proof of the topological Riemann–Roch formula (6.42) is relatively straightforward and purely algebraic topological, while the proof of (I.6) requires a considerable effort and relies on a heavy use of analysis.

REMARK I.6. More explicitly, Theorem I.4 states that

$$\chi(\exp \xi) = \int_M e^{c_1^G(\mathbb{L})(\sqrt{-1}\xi)} \operatorname{Td}^G(TM)(\sqrt{-1}\xi). \tag{I.7}$$

Note that the right-hand side of (I.7) is a germ of an analytic function on $\mathfrak{g}$, evaluated on ξ. (When $\mathfrak{g}$ is identified with $\mathbb{R}^n$, one has to plug $\sqrt{-1}\xi$ in the right-hand side of (I.7) rather than ξ.) This identity should be treated with care. The integrand on the right-hand side of (I.7) is not well defined; in general, a non-zero element $\xi \in \mathfrak{g}$ cannot be plugged in a cohomology class. However, the integral in (I.7), when evaluated at ξ is well defined as pointed out in Remark C.56. For (I.7) to literally make sense, one should replace the cohomology class of the integrand by an equivariant differential form representing it. (See Remark C.56.)

Applying the Atiyah–Bott–Berline–Vergne localization theorem (see Theorem C.53) to the integrand of (I.6) (or (I.7)), one gets a localized version of the equivariant localization theorem, due to Atiyah–Segal–Singer, [**ASe, ASi**], as follows. Let F be a connected component of M^G and let $j\colon F \to M$ be the inclusion map. Denote by TM and TF the stable tangent bundles to M and F, respectively. Note that since M carries a G-equivariant stable complex structure, these are complex vector bundles. By the splitting principle we assume that j^*TM splits equivariantly into a sum of line bundles

$$j^*TM = \mathbb{L}_1 \oplus \ldots \oplus \mathbb{L}_n. \tag{I.8}$$

We can also assume that the first r summands span the normal bundle of F and the remaining $n-r$ the tangent bundle of F. Let α_i be the weight of the isotropy representation of G on $\mathbb{L}_i$. Furthermore, the weight of the isotropy representation of G on the fibers of $\mathbb{L}$ over F is the constant value $\Phi_F = \Phi(F)$.

THEOREM I.7 (The localized equivariant index theorem, I). *On an open and dense subset of $\mathfrak{g}$, where the right-hand side is defined,*

$$\chi \circ \exp = \sum_F e^{\Phi_F/2\pi} \int_F e^{c_1(j^*\mathbb{L})} \operatorname{Td}(TF) \prod_{i=1}^r \left(1 - e^{-\alpha_i/2\pi} e^{c_1(\mathbb{L}_i)}\right)^{-1}. \tag{I.9}$$

PROOF. From the Atiyah–Bott–Berline–Vergne localization theorem and Theorem I.4 we conclude that

$$\chi \circ \exp = \sum_F \int_F \frac{e^{c_1^G(j^*\mathbb{L})} \operatorname{Td}^G(j^*TM)}{\prod_{i=1}^r c_1^G(\mathbb{L}_i)}. \tag{I.10}$$

By the product formula for the Todd class (Theorem I.1),

$$\operatorname{Td}^G(j^*TM) = \operatorname{Td}(TF) \prod_{i=1}^r \frac{c_1^G(\mathbb{L}_i)}{1 - e^{-c_1^G(\mathbb{L}_i)}}.$$

By Lemma I.3,

$$j^* c_1^G(\mathbb{L}_i) = c_1(\mathbb{L}_i) - \frac{1}{2\pi}\alpha_i(\xi)$$

for $1 \le i \le r$ and $j^* c_1^G(\mathbb{L}_i) = c_1(\mathbb{L}_i)$ for $r < i \le d$. Furthermore,

$$c_1^G(j^*\mathbb{L}) = c_1(\mathbb{L}) - \frac{1}{2\pi}\Phi_F.$$

After some cancelations (I.10) becomes (I.9). By analytic continuation, Theorem I.7 follows from Theorem I.4. □

Formula (I.9) holds only over an open and dense subset of $\mathfrak{g}$. One can, however, extend it to all of $\mathfrak{g}$ with the sum now depending on ξ. (This is the standard form of the localized equivariant index theorem.) Namely, recall that M^ξ is by definition the submanifold $\{\xi_M = 0\}$. For connected components F of M^ξ we still assume that the splitting (I.8) exists. Let us as usual identify $\mathfrak{g}$ with $\mathbb{R}^n$.

THEOREM I.8 (The localized equivariant index theorem, II). *For every $\xi \in \mathfrak{g}$,*

$$\chi(\exp\xi) = \sum_F e^{\sqrt{-1}\Phi_F(\xi)/2\pi} \int_F e^{c_1(j^*\mathbb{L})} \operatorname{Td}(TF) \prod_{i=1}^r \left(1 - e^{\sqrt{-1}\alpha_i(\xi)/2\pi} e^{-c_1(\mathbb{L}_i)}\right)^{-1}. \tag{I.11}$$

where F runs through the connected components of M^ξ.

For a generic ξ, we have $M^\xi = M^G$ and formula (I.11) turns into (I.9) evaluated at ξ.

The proof of this theorem follows the same line as the proof of Theorem I.7, but Theorem C.57 is used in place of Theorem C.53.

Remark I.9. Theorems I.4 and I.8 are stated here in the order opposite of how these results are proved. Theorem I.8 is usually proved first, and Theorem I.4 is derived from it.

In the next section we will discuss a version of the equivariant index theorem for *discrete* groups. In order to see how this is related to Theorem I.8, we will rewrite the first factor in (I.11) in a slightly different form. For $g = \exp \xi$, let $\gamma_{\mathbb{L}}(g) = e^{\sqrt{-1}\Phi_F(\xi)/2\pi}$ and $\gamma_k(g) = e^{\sqrt{-1}\alpha_k(\xi)/2\pi}$. These functions are the characters of the isotropy representation of G on $\mathbb{L}\,|_F$ and on the line bundles $\mathbb{L}_k$, respectively. Theorem I.8 immediately implies

Corollary I.10.

$$\chi(g) = \sum_F \gamma_{\mathbb{L}}(g) \int_F e^{c_1(j^*\mathbb{L})}\,\mathrm{Td}\,(TF) \prod_{k=1}^{r} \left(1 - \gamma_k(g) e^{-c_1(\mathbb{L}_k)}\right)^{-1}, \tag{I.12}$$

where F runs through the connected components of M^g.

Remark I.11. This follows from (I.11) when $M^g = M^\xi$, but is in fact true for all $g \in G$.

Remark I.12. It is clear that the products in (I.9), (I.11), and (I.12) can be expressed purely in terms of the equivariant (but not ordinary) Chern classes of the normal bundles $\mathcal{V}_F$ to the connected components F. In this form, Theorems I.7, I.8, and Corollary I.10 do not rely on the assumption that the bundles $\mathcal{V}_F$ split. The role of the splitting in these results is, however, more serious than a mere technical simplification of the proofs. The advantage of the statements given above is that in (I.9), (I.11), and (I.12) the topology of $\mathcal{V}_F$ is separated from the G-action on $\mathcal{V}_F$. This feature is inevitably lost when the splitting is not assumed.

3. The Kawasaki Riemann–Roch formula I: finite abelian quotients

In [**ASi**] Atiyah and Singer discuss an equivariant index theorem for *finite* groups. If Γ is a finite *abelian* group, this theorem reads

Theorem I.13. *Let Γ be an abelian group of symmetries of the* Spin^c*-Dirac operator, and let $\chi \colon \Gamma \to \mathbb{C}$ be the character of the virtual representation $\mathcal{Q}(M)$. Then, for $g \in \Gamma$, the value $\chi(g)$ is given by* (I.12).

In this section we will derive from this result the Kawasaki Riemann–Roch formula for the orbifold M/Γ. This formula will be, in some sense, a "trivial" case of the Kawasaki Riemann-Roch theorem since most orbifolds do not have global presentations of the form M/Γ with Γ finite. However, the Kawasaki Riemann–Roch theorem is an example of a theorem which is "almost as hard to state as to prove", and all the complications involved in the statement of the theorem are already in evidence for the orbifold M/Γ. In fact, the simplest formulation of the Kawasaki Riemann–Roch theorem may be interpreted as the assertion that "the index theorem for an arbitrary orbifold is identical with the index theorem for orbifolds of the form M/Γ."

To compute the stable complex quantization of M/Γ with no group acting on M/Γ, we must first define this quantization and this we will do by simply defining it to be the dimension of the virtual vector space $\mathcal{Q}(M)^\Gamma$. In other words, we will define it by assuming that quantization and reduction commute, *i.e.*,

$$\mathcal{Q}(M/\Gamma) = \mathcal{Q}(M)^\Gamma.$$

By Frobenius' theorem (see, e.g., [**Ki2**]),

$$\dim \mathcal{Q}(M)^{\Gamma} = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi(g). \tag{I.13}$$

In principle, we already know how to compute all the summands in this formula by Corollary I.10. In what follows we will do this computation carefully, so as to make the "bookkeeping" involved as simple as possible.

To this end, consider the orbit type stratification of M (see Appendix B) and denote by $\mathcal{S}$ the set of its orbit type strata. For each each $S \in \mathcal{S}$ the stabilizer group Γ_p with $p \in S$, is independent of p since Γ is abelian. We will denote this group by Γ_S and call it the *isotropy group* of the stratum S. In addition, we will denote by $\Gamma_S^{\#}$ the set of elements $g \in \Gamma_S$ with the property that for some Γ_S-invariant neighborhood U of the closure $F = \overline{S}$ of S, we have $U^g = \overline{S}$. In other words, $\Gamma_S^{\#}$ is the set of $g \in \Gamma$ for which the closure F is a connected component of M^g. (Note that $\Gamma_S^{\#}$ is not, in general, a subgroup. For example, $e \in \Gamma_S^{\#}$ if and only if S is the open and dense stratum. For this stratum we have $\Gamma_S^{\#} = \{e\}$ and $\Gamma_S = \Gamma$.) Clearly, $\Gamma_S^{\#} \subset \Gamma_S$.

We have a natural bijection

$$\{(S, g) \mid s \in \mathcal{S},\ g \in \Gamma_S^{\#}\} \longleftrightarrow \{(g, F) \mid g \in \Gamma,\ F \text{ a component of } M^g\}$$

obtained by $(S, g) \mapsto (g, F = \overline{S})$.

We can now state a preliminary version of the Kawasaki Riemann-Roch theorem for M/Γ. For $S \in \mathcal{S}$ let $j\colon F = \overline{S} \to M$ be the inclusion map and $\mathcal{V}_F = j^*TM/TF$ the normal bundle of F. Note that here TM and TF are stable tangent bundles, but $\mathcal{V}_F$ is the actual normal bundle to F in M. Furthermore, $\mathcal{V}_F$ is an equivariant complex bundle over F. By the splitting principle, we assume that $\mathcal{V}_F$ splits equivariantly into a sum of line bundles:

$$\mathcal{V}_F = \mathbb{L}_1 \oplus \ldots \oplus \mathbb{L}_r.$$

Let γ_i be the character of the representation of Γ_S on $\mathbb{L}_i$ and $\gamma_{\mathbb{L}}$ the character of the representation of Γ_S on $\mathbb{L}\,|_F$. Also let

$$\operatorname{Ind}(S) = \sum_{g \in \Gamma_S^{\#}} \gamma_{\mathbb{L}}(g) \int_F e^{c_1(j^*\mathbb{L})} \operatorname{Td}(TF) \prod_{i=1}^{r} \Big(1 - \gamma_i(g) e^{-c_1(\mathbb{L}_i)}\Big)^{-1}. \tag{I.14}$$

Observe that Γ acts on $\mathcal{S}$ and that $\operatorname{Ind}(S) = \operatorname{Ind}(gS)$. Finally, for $S \in \mathcal{S}$ denote by Γ^S the subgroup of all $g \in \Gamma$ which fix S, i.e., $gS = S$. Since Γ is abelian, $\Gamma_S = \Gamma_{gS}$ and $\Gamma^S = \Gamma^{gS}$ for all $S \in \mathcal{S}$ and $g \in \Gamma$. As a result, $\operatorname{Ind}(Y)$ and the subgroups Γ_Y and Γ^Y are well defined for an equivalence class $Y \in \mathcal{S}/\Gamma$.

PROPOSITION I.14.

$$\dim \mathcal{Q}(M/\Gamma) = \sum_{Y \in \mathcal{S}/\Gamma} \frac{1}{|\Gamma^Y|} \operatorname{Ind}(Y). \tag{I.15}$$

PROOF. First note that

$$\dim \mathcal{Q}(M/\Gamma) = \frac{1}{|\Gamma|} \sum_{S \in \mathcal{S}} \operatorname{Ind}(S). \tag{I.16}$$

This can be easily proved by expressing $\dim \mathcal{Q}(M/\Gamma)$ as a double sum using (I.13) and (I.12) and comparing the result with the double sum given by (I.16) and (I.14). We leave the details to the reader as an exercise.

Since $\operatorname{Ind}(S)$ is independent of S representing a given class Y, each $\operatorname{Ind}(Y)$ occurs on the right-hand side of (I.16) exactly $[\Gamma : \Gamma^Y]$ times. Thus the right-hand side of (I.16) is equal to the right-hand side of (I.15). □

Our final version of the Kawasaki Riemann-Roch theorem for M/Γ consists simply of interpreting (I.15) in the language of orbifolds. The orbit type stratification of M descends to a stratification of M/Γ indexed by the poset $\mathcal{S}/\Gamma$. (This stratification is called the *orbifold stratification* of M/Γ). Recall that the strata of this stratification are

$$X = \left(\bigsqcup_{g \in \Gamma} gS \right) /\Gamma = S/\Gamma^S = S/(\Gamma^S/\Gamma_S).$$

In particular, all S representing the same class $Y \in \mathcal{S}/\Gamma$ project to the same X. (Note that Γ^S does not act freely on S but Γ^S/Γ_S does and hence X is a smooth manifold.) The closure $\overline{X}$ of X in M/Γ is the orbifold F/Γ^S.

For each of the orbifolds $\overline{X}$, consider the integral on the right-hand side of (I.12). The integrand is represented by a Γ^S-invariant differential form, and so it can be regarded as an (orbifold) form on the quotient space $\overline{X} = F/\Gamma^S$. Dividing the integral by $|\Gamma^S|$, we get the integral in the orbifold sense of this form over the (non-effective) orbifold $\overline{X}$. Denote the sum of the resulting orbifold integrals by $\operatorname{Ind}(X)$, so that $\operatorname{Ind}(X) = \operatorname{Ind}(Y)/|\Gamma^Y|$. We will refer to $\operatorname{Ind}(X)$ as the "local Kawasaki index" of the stratum X. From Proposition I.14, we immediately obtain

Corollary I.15.

$$\dim \mathcal{Q}(M/\Gamma) = \sum \operatorname{Ind}(X), \tag{I.17}$$

where the sum extends over the strata of the orbifold stratification of M/Γ.

Remark I.16. Frobenius' theorem can also be used to obtain a formula for the ordinary Euler characteristic of M/Γ:

$$\chi(M/\Gamma) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \chi(M^g), \tag{I.18}$$

where Γ is a finite, but not necessarily abelian, group acting on M. Let us outline the proof of (I.18); see also [**Sh**]. Observe first that $H^*(M/\Gamma) = H^*(M)^\Gamma$ by Corollary B.36. (Here the cohomology is taken with real coefficients and since Γ is discrete basic forms are just Γ-invariant forms.) Hence,

$$\chi(M/\Gamma) = \sum (-1)^k \dim H^k(M)^\Gamma.$$

By applying Frobenius' theorem as in (I.13), we see that

$$\chi(M/\Gamma) = \frac{1}{|\Gamma|} \sum_{g \in \Gamma} \operatorname{trace}(g^*), \tag{I.19}$$

where $\operatorname{trace}(g^*)$ stands for the trace of the linear operator g^* on the virtual representation $\sum (-1)^k \dim H^k(M)$. These traces can be found using the Lefschetz formula as follows. For a diffeomorphism $f \colon M \to M$ with non-degenerate isolated fixed points the Lefschetz formula asserts that $\operatorname{trace}(f^*) = \sum_{x \in M^G} \sigma_x(f)$. Here

$\sigma_x(f)$ is the parity of the number of real eigenvalues of $df\colon T_xM \to T_xM$ which are greater than one. By taking a small, tangent to M^G, perturbation of $g\colon M \to M$, we conclude that $\mathrm{trace}(g^*) = \sum_F \sigma_F(g)\chi(F)$, where F runs through connected components of M^G and $\sigma_F(g)$ is the parity of the number of real eigenvalues of dg restricted to a fiber of $\mathcal{V}_F$. Since Γ is compact, dg has no real eigenvalues greater than one and $\sigma_F(g) = 1$. As a result,

$$\mathrm{trace}(g^*) = \sum_F \chi(F) = \chi(M^g).$$

Combining this formula with (I.19), we obtain (I.18).

4. The Kawasaki Riemann-Roch formula II: torus quotients

4.1. The Kawasaki Riemann-Roch formula. In this section we will describe how to compute the dimension of the quantization space $\mathcal{Q}(Z/G)$ for orbifolds which have presentations of the form Z/G, where G is a compact connected abelian Lie group acting on a compact manifold Z in a locally free fashion. We will see that as far as "bookkeeping" is concerned, this computation is virtually identical with the computation of $\dim \mathcal{Q}(M/\Gamma)$ carried out in the previous section.

Consider, as in the previous section, the orbit type stratification of Z. (See Appendix B for the definition.) In what follows we keep the notation from Section 3. Thus, $\mathcal{S}$ is the poset of the strata of the orbit type stratification; $G_S = G_p$ is the stabilizer of $p \in S \in \mathcal{S}$ (as before, it is independent of $p \in S$), and $G_S^\#$ is the set of all $g \in G_S$ for which the closure $F = \overline{S}$ is a connected component of Z^g. We will refer to G_S as the *isotropy subgroup* of S.

Note that since Z and G are compact, $\mathcal{S}$ is finite as are all subgroups G_S (see Appendix B). Furthermore, since G is connected, the G-action on Z preserves S. Hence, in the notation of the previous section, $G^S = G$ for any S. Moreover, it is clear that G/G_S acts *freely* on S and hence the quotient space S/G is a submanifold of Z/G. These submanifolds give rise to the orbifold stratification of Z/G. For each stratum S/G, its closure F/G in Z/G is a sub-orbifold of Z/G.

Let Z/G be equipped with a closed two-form ω_{red}, a pre-quantization line bundle $\mathbb{L}_{\mathrm{red}}$ and a stable complex structure. Via the projection $\pi\colon Z \to Z/G$ we can pull-back these to Z. As a result, we obtain a closed two-form $\omega = \pi^*\omega_{\mathrm{red}}$, a complex line bundle $\mathbb{L} = \pi^*\mathbb{L}_{\mathrm{red}}$, and a stable complex structure on Z.

Let us define, for each of the sub-orbifolds F/G, a local Kawasaki index as follows. Denote by $j\colon F \to Z$ the inclusion map. By the splitting principle, we assume that the restriction of the stable tangent bundle TZ to F splits equivariantly into a sum of complex line bundles

$$j^*TZ = \mathbb{L}_1 \oplus \ldots \oplus \mathbb{L}_m.$$

Without loss of generality, we can also assume that the first r summands on the right-hand side span the normal bundle of F. Let us denote by γ_i, for $i = 1, \ldots, m$, the character of the representation of Γ_F on $\mathbb{L}_i$. Note that $\gamma_i \equiv 1$ for $i = r+1, \ldots, m$ and

$$\gamma_i(g) \neq 1 \quad \text{for all} \quad i = 1, \ldots, r \tag{I.20}$$

if and only if $g \in G_S^\#$. We also denote by $\gamma_{\mathbb{L}}$ the character of the representation of G_S on the line bundle $\mathbb{L}$.

Since G acts on Z (and hence on F) in a locally free fashion, the map

$$\pi^*\colon H^*(F/G) \to H^*_G(F)$$

is an isomorphism. (See Appendix C.) Therefore, for each of the equivariant Chern classes $c_1^G(\mathbb{L}_i) \in H^2_G(F)$ there exists an element $u_i \in H^2(F/G)$ such that $\pi^* u_i = c_1^G(\mathbb{L}_i)$. In addition, there exists an orbifold Chern class $c_1(\mathbb{L}_{\text{red}}) \in H^2(F/G)$ associated with the orbifold line bundle $\mathbb{L}_{\text{red}}$ for which we have $\pi^* c_1(\mathbb{L}_{\text{red}}) = c_1^G(\mathbb{L})$ since $\mathbb{L} = \pi^*\mathbb{L}_{\text{red}}$.

Now we are in a position to define the local Kawasaki index of the orbifold F/G to be the integral

$$\text{(I.21)} \qquad \operatorname{Ind}(F/G) = \int_F \tau_F^{\text{nor}} \tau_F^{\tan},$$

where

$$\text{(I.22)} \qquad \tau_F^{\text{nor}} = \sum_{g \in G_F^\#} \gamma_{\mathbb{L}}(g) \prod_{i=1}^r \left(1 - \gamma_i(g) e^{-u_i}\right)^{-1}$$

and

$$\tau_F^{\tan} = e^{c_1(\mathbb{L}_{\text{red}})} \prod_{i=r+1}^m \frac{u_i}{1 - e^{-u_i}}.$$

Remark I.17. The inverses $(1 - \gamma_i(g) e^{-u_i})^{-1}$ in (I.22) are indeed defined. To show this, we first note that

$$\gamma_i(g) e^{-u_i} = \gamma_i(g) + \dots$$

with the dots indicate cohomology classes of degree greater than zero. By (I.20), $\gamma_i(g) \neq 1$. Hence the expansion of $(1 - \gamma_i(g) e^{-u_i})^{-1}$ does not have $1/0$ as its leading term. Also notice that if we deleted the normal contribution τ_F^{nor} from the integrand in (I.21) we would be just computing the "classical" Riemann–Roch number of F/G. In particular, this is exactly what $\operatorname{Ind}(F/G)$ is for the largest stratum, i.e., for $F = Z/G$.

The definitions of quantization given in Chapter 6 make perfectly good sense for orbifolds. (See, in particular, Section 7 of Chapter 6.) In fact, for orbifolds of the form Z/G the Dolbeault and Spin$^\text{c}$-Dirac operators can be defined on Z by means of the pre-quantization structure and the complex structure on Z. This operator will not be elliptic, but it will be *transversally* elliptic in the sense of Atiyah (see, e.g., [**At1, Ve3, BV3, BV4**]). Thus the dimension of $\mathcal{Q}(Z/G)$ can, in this case, be computed by using, instead of the Kawasaki Riemann–Roch formula, a somewhat simpler index theorem for transversally elliptic operators proved by Atiyah in [**At1**]. We will not attempt to give here a precise description of Atiyah's result since we are not concerned with the details of the proof of the Kawasaki theorem, but only with the details of the *statement* of this theorem. For orbifolds with presentations of the form Z/G the Kawasaki Riemann–Roch theorem asserts:

Theorem I.18. *Let $\mathcal{Q}(Z/G)$ be the* Spin$^\text{c}$ *quantization of Z/G. Then*

$$\text{(I.23)} \qquad \dim \mathcal{Q}(Z/G) = \sum_{S \in \mathcal{S}} \operatorname{Ind}(F/G),$$

where $F = \overline{S}$ and $\operatorname{Ind}(F/G)$ *is defined by* (I.21).

We will call the terms on the right-hand side of (I.23) the *Kawasaki invariants* of Z/G.

4.2. Cobordism invariance of geometric quantization. In this section we show that the geometric quantization of a reduced space is an invariant of cobordism.

Consider a Hamiltonian G-manifold (M, ω, Φ) equipped with a G-equivariant stable complex structure J, for G a torus. Here and in what follows we assume that Φ is proper. Let $c \in \mathbb{Z}_G^*$ be a regular value of Φ. Then the quotient space $M_{\text{red}} = \Phi^{-1}(c)/G$ is a stable complex oriented orbifold (See Section 2.3 of Chapter 5 for a discussion of the reduction of stable complex structures). Furthermore, the form ω descends to this orbifold. Hence, we have sufficient data to define the stable complex quantization $\mathcal{Q}(M_{\text{red}})$.

Next, let $(M_0, \omega_0, \Phi_0, J_0)$ and $(M_1, \omega_1, \Phi_1, J_1)$ be two such Hamiltonian G-manifolds which are cobordant via a G-equivariant stable complex G-manifold with proper moment map Φ in the sense of Chapter 2. Let $c \in \mathbb{Z}_G^*$ be a regular value of both Φ_0 and Φ_1. As before, the geometric quantizations $\mathcal{Q}((M_0)_{\text{red}})$ and $\mathcal{Q}((M_1)_{\text{red}})$ are defined.

THEOREM I.19. *Let c be a regular value of Φ. Then $\mathcal{Q}((M_0)_{\text{red}}) = \mathcal{Q}((M_1)_{\text{red}})$.*

PROOF. Since c is a regular value of Φ, the cobordism between $(M_0, \omega_0, \Phi_0, J_0)$ and $(M_1, \omega_1, \Phi_1, J_1)$ gives rise to an orbifold cobordism of oriented orbifolds $(M_0)_{\text{red}}$ and $(M_1)_{\text{red}}$ (see Chapter 5). The stable complex structure descends to this cobordism as does the cohomology class of the equivariant two-form. Similarly, we have orbifold cobordisms between closures of strata of $(M_0)_{\text{red}}$ and $(M_1)_{\text{red}}$. It is an immediate consequence of Stokes' formula and (I.21) that $\text{Ind}(F_0/G) = \text{Ind}(F_1/G)$, where F_0/G and F_1/G are cobordant closures of strata in $(M_0)_{\text{red}}$ and $(M_1)_{\text{red}}$, respectively. The theorem follows now from (I.23). Alternatively, the cobordism invariance of the index can be proved directly as in Appendix J. □

In a similar vein one can use the Kawasaki Riemann–Roch theorem to prove the quantum shift formula (Proposition 6.49). The argument requires a careful (and non-trivial) calculation which equates the Kawasaki invariant for a stratum computed for J_0 and $\mathbb{L}$ with that computed for J_1 and $\mathbb{L} \otimes \mathbb{L}_\delta$.