Stokes' Theorem

Parametric Surfaces

Just as a curve can be described as a vector function $\bar{r}(t)$ of a single parameter t, a surface can be described by a vector function $\bar{r}(u,v)$ of two parameters u and v, that is

$$\bar{r}(u,v) = x(u,v)\bar{i} + y(u,v)\bar{j} + z(u,v)\bar{k}$$

is a vector-valued function defined on a region D in the uv-plane. The component functions x, y, and z of $\bar{r}(u,v)$ are functions of two variables u and v with domain D. The set of all points (x,y,z) in R^3 such that

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v) \qquad (1)$$

and (u,v) varies throughout D is called a parametric surface S. The parametric equations of S are (1).

Normal Vector to a Surface

Say that a vector $\bar{v}$ is tangent to a surface S at the point P if $\bar{v}$ is a tangent vector, at P, to some curve that is contained in S. This is analogous to the tangent line of a single variable function.

Assume that a surface S is represented by $\bar{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v) \in D \subseteq \mathrm{R}^2$. At a point on the surface (u_0,v_0), there exists the following two tangent vectors, $\bar{T}_u(u_0,v_0)$ and $\bar{T}_v(u_0,v_0)$, given by

$$\bar{T}_u(u_0,v_0) = \left\langle \left.\frac{\partial x}{\partial u}\right|_{(u_0,v_0)}, \left.\frac{\partial y}{\partial u}\right|_{(u_0,v_0)}, \left.\frac{\partial z}{\partial u}\right|_{(u_0,v_0)} \right\rangle,$$

being the partial derivatives of the components $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$ with respect to u, and

$$\bar{T}_v(u_0,v_0) = \left\langle \left.\frac{\partial x}{\partial v}\right|_{(u_0,v_0)}, \left.\frac{\partial y}{\partial v}\right|_{(u_0,v_0)}, \left.\frac{\partial z}{\partial v}\right|_{(u_0,v_0)} \right\rangle,$$

being the partial derivatives of the components $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$ with respect to v.

The vector $\bar{N}(u_0,v_0) = \bar{T}_u(u_0,v_0) \times \bar{T}_v(u_0,v_0)$, being perpendicular to both $\bar{T}_u(u_0,v_0)$ and $\bar{T}_v(u_0,v_0)$ is called a **normal vector**. The normal vector is computed by

$$\bar{N} = \bar{T}_u \times \bar{T}_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

where we have dropped explicit reference to the point (u_0,v_0).

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise boundary curve C with positive orientation. Let $\bar{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. Then

$$\int_{\mathrm{C}} \bar{F} \cdot d\bar{r} = \iint_{\mathrm{S}} \mathrm{curl}\bar{F} \cdot d\bar{S}.$$

The positively oriented boundary curve of the oriented surface S is often written as $\partial\mathrm{S}$. Then, Stokes' Theorem can be expressed as

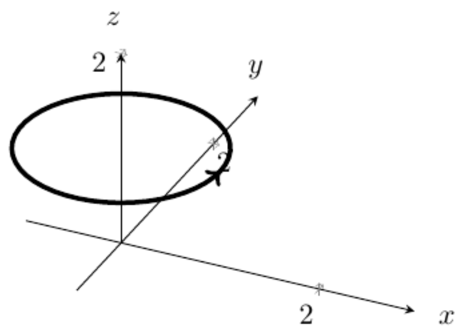
$$\iint_{\mathrm{S}} \mathrm{curl}\bar{F} \cdot d\bar{S} = \int_{\partial\mathrm{S}} \bar{F} \cdot d\bar{r}.$$

The left side of the equation involves an integral involved with derivatives because of $\mathrm{curl}\bar{F}$, and the right side has the values of $\bar{F}$ at the boundary of S.

In a special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is $\bar{N}$, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \overline{F}\cdot d\overline{r} = \iint_S \operatorname{curl}\overline{F}\cdot d\overline{S} = \iint_S \operatorname{curl}\overline{F}\cdot \overline{N}dA.$$

Example 1. Let C be the circle $x^2+y^2=1$ and $z=1$, oriented counterclockwise as seen from a point $(0,0,z)$ with $z>1$ on the z-axis. For $\overline{F}=2\overline{i}+x\overline{j}+y^2\overline{k}$, compute $\int_C \overline{F}\cdot d\overline{r}$.



Solution. Calculating directly, we parametrize C by $\overline{r}=\langle \cos t,\sin t,1\rangle$ where $t\in[0,2\pi)$. Then $\overline{r'}(t)=\langle -\sin(t),\cos(t),0\rangle$ We then obtain

$$\begin{aligned}
\int_C \overline{F}\cdot d\overline{r} &= \int_C \overline{F}\left(\overline{r}(t)\right)\cdot \overline{r}(t)\,dt \\
&= \int_0^{2\pi} \left\langle 2,\cos(t),\sin^2(t)\right\rangle\cdot\left\langle -\sin(t),\cos(t),0\right\rangle dt \\
&= \int_0^{2\pi} \left(-2\sin(t)+\cos^2(t)\right)dt \\
&= \int_0^{2\pi} \left(-2\sin(t)+\frac{1}{2}+\frac{1}{2}\cos(2t)\right)dt \\
&= \left[2\cos(t)+\frac{1}{2}t+\frac{1}{4}\sin(2t)\right]_0^{2\pi} = \pi,
\end{aligned}$$

using the fact that $\cos^2(t)=\frac{1}{2}+\frac{1}{2}\cos(2t)$.

Alternatively, we can solve this problem using Stokes' Theorem:

$$\int_C \overline{F}\cdot d\overline{r} = \iint_S \operatorname{curl}\overline{F}\cdot d\overline{S}$$

We first calculate the curl:

$$\text{curl}\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2 & x & y^2 \end{vmatrix} = 2y\bar{i} + \bar{k}.$$

The surface bounded by C is the disk given by $x^2 + y^2 \leq 1$ and $z = 1$. We can parameterize this surface by $\bar{r}(u,v) = \langle u, v, 1\rangle$, for $u^2 + v^2 \leq 1$. We can then calculate the tangent and normal vectors:

$$\bar{T}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle = \langle 1, 0, 0\rangle,$$

$$\bar{T}_y = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle = \langle 0, 1, 0\rangle,$$

$$\bar{N} = \bar{T}_u \times \bar{T}_v = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \bar{k}.$$

Now, we have shown the surface is flat and lies in the xy-plane with upward orientation, so Stokes' Theorem becomes

$$\int_{\mathrm{C}} \bar{F} \cdot d\bar{r} = \iint_{\mathrm{S}} \text{curl}\bar{F} \cdot d\bar{S} = \iint_{\mathrm{S}} \text{curl}\bar{F} \cdot \bar{N} dA.$$

Hence, we have

$$\int_{\mathrm{C}} \bar{F} \cdot d\bar{r} = \iint_{\mathrm{S}} \text{curl}\bar{F} \cdot d\bar{S} = \iint_{u^2+v^2\leq 1} \left(2y\bar{i} + \bar{k}\right) \cdot \bar{k} dA = \iint_{u^2+v^2 \leq 1} 1 dA = \pi.$$

Example 2. Let S be the interior of the unit sphere, i.e. $x^2 + y^2 + z^2 \leq 1$. Compute $\iiint_{\mathrm{S}} \text{curl}\bar{F} \cdot d\bar{S}$, where $\bar{F}(x,y,z) = z\bar{i} + y\bar{j} + x\bar{k}$.

Solution. We use spherical coordinates to write $\bar{F}$ in terms of two variables.

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta, \qquad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi,$$

where $r = 1$. The parametric representation of the surface $x^2 + y^2 + z^2 = 1$ is given by

$$\bar{r}(\theta,\phi) = \sin\theta\cos\phi\bar{i} + \sin\theta\sin\phi\bar{j} + \cos\theta\bar{k}$$

and the tangent and normal vectors are given by

$$\overline{T}_\theta = \frac{\partial x}{\partial \theta}\overline{i} + \frac{\partial y}{\partial \theta}\overline{j} + \frac{\partial z}{\partial \theta}\overline{k} = \cos\theta\cos\phi\overline{i} + \cos\theta\sin\phi\overline{j} - \sin\theta\overline{k},$$

$$\overline{T}_\phi = \frac{\partial x}{\partial \phi}\overline{i} + \frac{\partial y}{\partial \phi}\overline{j} + \frac{\partial z}{\partial \phi}\overline{k} = -\sin\theta\sin\phi\overline{i} + \sin\theta\cos\phi\overline{j},$$

$$\overline{N} = \overline{T}_\theta \times \overline{T}_\phi = \sin^2\theta\cos\phi\overline{i} + \sin^2\theta\sin\phi\overline{j} + \sin\theta\cos\theta\overline{k}.$$

Finally, applying Stokes' Theorem, we obtain

$$\iiint_S \mathrm{curl}\overline{F}\cdot d\overline{S} = \iint_{\partial S}\overline{F}\cdot d\overline{r} = \iint_{\partial S}\overline{F}\cdot\overline{N}dA$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\cos\theta\overline{i} + \sin\theta\sin\phi\overline{j} + \sin\theta\cos\phi\overline{k}\right)\cdot\left(\sin^2\theta\cos\phi\overline{i} + \sin^2\theta\sin\phi\overline{j} + \sin\theta\cos\theta\overline{k}\right)d\theta d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(2\sin^2\theta\cos\theta\cos\phi + \sin^3\theta\sin^2\phi\right)d\theta d\phi$$

$$= 2\int_0^{\pi}\sin^2\theta\cos\theta d\theta\int_0^{2\pi}\cos\phi d\phi + \int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}\sin^2\phi d\phi$$

$$= 2\int_0^{\pi}\sin^2\theta\cos\theta d\theta\left[\sin\phi\right]_0^{2\pi} + \int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta d\theta\int_0^{2\pi}\frac{1}{2}\left(1-\cos\left(2\phi\right)\right)d\phi$$

$$= 0 - \int_1^{-1}\left(1-u^2\right)du \times \left[\frac{1}{2}t - \frac{1}{4}\sin\left(2\phi\right)\right]_0^{2\pi}$$

$$= \left[u - u^3\right]_{-1}^{1} \times \pi = \frac{4}{3}\pi,$$

where we use the u-substitution $u = \cos\theta$ and the fact that $\sin^2(t) = \dfrac{1+\cos(2t)}{2}$.