Series

An expression of the form \( a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=1}^{\infty} a_n \) is called an infinite series or simply a series, where \( a_n \) is the \( n \)th term of a sequence.

The \( n \)th partial sum \( s_n \) of an infinite series \( \sum_{n=1}^{\infty} a_n \) is \( s_n = a_1 + a_2 + \ldots + a_n \).

Definition: An infinite series \( \sum_{n=1}^{\infty} a_n \) is convergent if \( \lim_{n \to \infty} s_n = s \) for some real number \( s \). The series is divergent if the limit does not exist (i.e., if it is not a real number).

If the infinite series \( \sum_{n=1}^{\infty} a_n \) is convergent and \( \lim_{n \to \infty} s_n = s \), then \( s \) is called the sum of the series and can be written as \( s = a_1 + a_2 + \ldots + a_n + \ldots \). If the series diverges, then it has no sum.

Geometric Series

The geometric series, \( \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots = \sum_{n=0}^{\infty} ar^n \), converges if \( |r| < 1 \) and has the sum \( \frac{a}{1-r} \) where \( a \) is the first term of the series and \( r \) is called the common ratio. The geometric series diverges if \( |r| \geq 1 \).

Example. Determine if the series converges or diverges. If the series converges, find its sum.
Solution. (a) Rewrite the series in the form: $\sum_{n=1}^{\infty} ar^{n-1}$

\[
\sum_{n=1}^{\infty} 5^{-n} 2^{n+1} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} = \sum_{n=1}^{\infty} \frac{2^{n+1}}{5^n} \cdot \frac{5}{5} = \sum_{n=1}^{\infty} \frac{2}{5}^{n-1}
\]

Then, $a = \frac{4}{5}$ and $r = \frac{2}{5} < 1$. Thus, this geometric series converges, and its sum is

\[
\frac{a}{1-r} = \frac{\frac{4}{5}}{1-\frac{2}{5}} = \frac{4}{3}.
\]

(b) Rewrite the series in the form: $\sum_{n=0}^{\infty} ar^n$

\[
\sum_{n=0}^{\infty} \frac{(-2)^{2n}}{3^{n-1}} = \sum_{n=0}^{\infty} 3^{n} \cdot \frac{4^n}{3^n} = \sum_{n=0}^{\infty} 3 \left(\frac{4}{3}\right)^n
\]

Then, $r = \frac{4}{3} > 1$. Thus, this geometric series diverges, and it has no sum.

**Telescoping Series**
Use partial fraction decomposition for rational expressions to calculate the partial sum; for instance, to find the partial sum of \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \), use \( \frac{1}{n} - \frac{1}{n+1} \). Then, expand the series up to its nth term to find the partial sum.

Example. Prove that the following infinite series converges, and find its sum:

\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{n(n+1)} + \ldots
\]

Solution. The partial sum of this infinite series is

\[
S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \ldots + \frac{1}{n(n+1)}
\]

Using partial fraction decomposition on the rational expression of the partial sum: \( \frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1} \)

Then, once expanding the sum, the partial sum reduces to \( n \)

\[
S_n = \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+1} \right) = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots
\]

\[
+ \left( \frac{1}{n-2} - \frac{1}{n-1} \right) + \left( \frac{1}{n-1} - \frac{1}{n} \right) + \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]

\[
= 1 - \frac{1}{n+1}
\]

Finally, take the limit of the partial sum as \( n \) approaches to \( \infty \).

\[
\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = \lim_{n \to \infty} \left( 1 - \frac{1/n}{1 + 1/n} \right) = 1.
\]

p-series
Fact: The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series) is divergent since it is a p-series with $p = 1$. 