A General Procedure for Sketching the Graph of a Function

Step 1. Find the domain of \( y = f(x) \) (that is, all \( x \) values where \( f(x) \) is defined).

Step 2. Find and plot all intercepts. The \( y \)-intercept (where \( x = 0 \)) is usually easy to find, but the \( x \)-intercept (where \( y = f(x) = 0 \)) may be difficult to find as it involves solving an equation.

Step 3. Determine all vertical and horizontal asymptotes of the graph of \( f(x) \). Draw the asymptotes by using dashed lines.

\[ \text{def: Vertical Asymptote} \]

The line \( x = c \) is a vertical asymptote of the graph of \( f(x) \) if either

\[ \lim_{x \to c^-} f(x) = +\infty \quad \text{(or } -\infty \text{)} \]

or \[ \lim_{x \to c^+} f(x) = +\infty \quad \text{(or } -\infty \text{)} \]

\[ \text{def: Horizontal Asymptote} \]

The horizontal line \( y = b \) is called a horizontal asymptote of the graph of \( y = f(x) \) if

\[ \lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b. \]

Step 4. Find \( f''(x) \) and use it to determine the critical numbers of \( f(x) \) and the intervals where \( f(x) \) is increasing and decreasing.

Procedure for Using the Derivative to Determine Intervals of Increase and Decrease for a Function \( f \)

Step I. Find the critical numbers, i.e., all values of \( x \) for which \( f''(x) = 0 \) or \( f''(x) \) does not exist, and mark these numbers on a number line. As well, mark the numbers which
are not in the domain of \( f(x) \). All these numbers divide the number line into open intervals.

Step II. Choose a test number \( c \) from each interval \( a < x < b \) determined in Step I., and evaluate \( f'(c) \). Then,

If \( f'(c) > 0 \), the function \( f(x) \) is increasing (graph rising) on \( a < x < b \).

If \( f'(c) < 0 \), the function \( f(x) \) is decreasing (graph falling) on \( a < x < b \).

Step 5. Determine the \( x \) and \( y \) coordinates of all relative extrema.

def: The First Derivative Test for Relative Extrema

Let \( c \) be a critical number for \( f(x) \) (that is, \( f(c) \) is defined and either \( f'(c) \) does not exist). Then the critical point \( P(c, f(c)) \) is

- a relative maximum - if \( f''(x) > 0 \) to the left of \( c \) and \( f'(x) < 0 \) to the right of \( c \)
- a relative minimum - if \( f''(x) < 0 \) to the left of \( c \) and \( f'(x) > 0 \) to the right of \( c \)
- not a relative extremum - if \( f'(x) \) has the same sign on both sides of \( c \)

Step 6. Find \( f''(x) \) and use it to determine intervals of concavity and points of inflection.

Second Derivative Procedure for Determining Intervals of Concavity for a Function \( f \)

Step I. Find all values of \( x \) for which \( f''(x) = 0 \) or \( f''(x) \) does not exist, and mark these numbers on a number line. As well, mark the numbers which are not in the domain of \( f(x) \). All these numbers divide the number line into open intervals.
Step II. Choose a test number \( c \) from each interval \( a < x < b \) determined in Step I., and evaluate \( f''(c) \). Then,

If \( f''(c) > 0 \), the graph of \( f(x) \) is concave upward on \( a < x < b \).

If \( f''(c) < 0 \), the graph of \( f(x) \) is concave downward on \( a < x < b \).

Procedure for Finding the Inflection Points of a Function \( f \)

Step I. Compute \( f''(x) \) and determine all points in the domain of \( f \) where either \( f''(c) = 0 \) or \( f''(c) \) does not exist.

Step II. For each number \( c \) found in Step I., determine the sign of \( f''(x) \) to the left and to the right of \( x = c \), that is, for \( x < c \), and for \( x > c \).

If \( f''(x) > 0 \) on one side of \( x = c \) and \( f''(x) < 0 \) on the other side, then \((c, f(c))\) is an inflection point of \( f \).

Step 7. Sketch by putting together all the information gathered from Steps 1-6. Be sure to remember that the graph cannot cross a vertical asymptote, but it can cross its horizontal asymptote, just not at negative infinity and positive infinity.

Example. Sketch the curve \( y = \frac{1}{x^3 - x} \).

Solution. 1. For \( y = \frac{1}{x^3 - x} = \frac{1}{x(x^2 - 1)} \), the domain consists of all \( x \) values except \( x = -1 \), \( x = 0 \), and \( x = 1 \).

2. There are no \( x \)-intercepts and \( y \)-intercepts.

3. \[
\lim_{x \to \pm \infty} \frac{1}{x^3 - x} = \lim_{x \to \pm \infty} \frac{1}{x^3} = 0
\]

Thus, the line \( y = 0 \) is a horizontal asymptote.
Since the denominator equals 0 when \(x = -1, x = 0,\) and \(x = 1\), and
\[
\lim_{x \to -1^+} \frac{1}{x^3 - x} = +\infty \quad \lim_{x \to -1^-} \frac{1}{x^3 - x} = -\infty
\]
\[
\lim_{x \to 0^+} \frac{1}{x^3 - x} = -\infty \quad \lim_{x \to 0^-} \frac{1}{x^3 - x} = +\infty
\]
\[
\lim_{x \to 1^+} \frac{1}{x^3 - x} = +\infty \quad \lim_{x \to 1^-} \frac{1}{x^3 - x} = -\infty
\]
The lines \(x = -1, x = 0,\) and \(x = 1\) are vertical asymptotes.

4. \(y' = \frac{-3x^2 + 1}{(x^3 - x)^2}\). Set \(y' = 0\). The solutions (i.e., the critical numbers) are \(x = \pm \frac{1}{\sqrt{3}}\).

Note that \(y'\) is not defined at \(x = -1, x = 0,\) and \(x = 1,\) but these are not critical numbers as they are not in the domain of \(f'(x)\). However, these \(x\) values must be included in the analysis of intervals where the function is increasing and decreasing.

Note that \(x = 1/\sqrt{3} \approx 0.58\) and \(x = -1/\sqrt{3} \approx -0.58\).

<table>
<thead>
<tr>
<th>dec</th>
<th>-1</th>
<th>-0.58</th>
<th>inc</th>
<th>0</th>
<th>inc</th>
<th>0.58</th>
<th>dec</th>
<th>1</th>
<th>dec</th>
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<tr>
<td>(f' &lt; 0)</td>
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<td>(f' &gt; 0)</td>
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The curve is increasing (inc) on \((-1/\sqrt{3}, 0)\) and \((0, 1/\sqrt{3})\). The curve is decreasing (dec) on \((-\infty, -1), (-1, -1/\sqrt{3}), (1/\sqrt{3}, 1),\) and \((1, \infty)\).
5. The critical numbers are \( x = \pm \frac{1}{\sqrt{3}} \).

The point \( \left( \frac{1}{\sqrt{3}}, -\frac{3\sqrt{3}}{2} \right) \) is a relative maximum.

The point \( \left( -\frac{1}{\sqrt{3}}, \frac{3\sqrt{3}}{2} \right) \) is a relative minimum.

6. \( y'' = \frac{2[6x^4 + 9x^2 + 1]}{(x^3 - x)^3} \)

Set \( y'' = 0 \). There are no solutions (thus, there are no inflection points). However, for the purpose of analyzing concavity, the numbers \( x = -1 \), \( x = 0 \), and \( x = 1 \) are considered which are not in the domain of \( f(x) \).

\[
\begin{array}{c|c|c|c|c}
& -1 & 0 & 1 \\
\hline
f'' < 0 & f'' > 0 & f'' < 0 & f'' > 0 \\
\end{array}
\]

The curve is concave up (CU) on \((-1, 0)\) and \((1, \infty)\) and is concave down (CD) on \((-\infty, -1)\) and \((0, 1)\).

7. Sketch of \( f(x) \).