Definition of a Derivative

Derivative of a Function $f(x)$ at a Number $a$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ or } f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ provided that the limit exists. (Note that the former formula is used more often than the latter.)}$$

Example. Using the definition of the derivative at a number, find the slope of tangent line at $x = -2$ to the curve $y = -6x^2$. What is the equation of the tangent line at this point?

Solution. By definition of the derivative, the slope of the tangent line at $x = -2$ is

$$f'(-2) = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$$

$$= \lim_{h \to 0} \frac{-6(-2+h)^2 - (-6(-2)^2)}{h}$$

$$= \lim_{h \to 0} \frac{-24 + 24h - 6h^2 + 24}{h}$$

$$= \lim_{h \to 0} \frac{h(24 - 6h)}{h} = \lim_{h \to 0} (24 - 6h) = 24$$

When $x = -2$, $y(-2) = -6(-2)^2 = -24$, that is, the point of tangency is $(-2, -24)$.

Using the point-slope formula,

$$y + 24 = 24(x + 2)$$

$$y = 24x + 24.$$
Derivative Function \( f'(x) \) of a Function \( f(x) \)

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \text{ for any } x \text{ for which this limit exists.}
\]

Example. Using the definition of the derivative, given the function \( f(x) = 3 - \frac{1}{x^2} \), find \( f'(x) \).

Solution. By the definition of the derivative,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{\left[ 3 - \frac{1}{(x + h)^2} \right] - \left[ 3 - \frac{1}{x^2} \right]}{h}
\]

\[
= \lim_{h \to 0} \frac{-x^2 + (x + h)^2}{h(x^2 + xh + h^2)} = \lim_{h \to 0} \left( \frac{-x^2 + x^2 + 2xh + h^2}{x^2(x + h)^2} \right) \left( \frac{1}{h} \right)
\]

\[
= \lim_{h \to 0} \frac{h(2x + h)}{hx^2(x + h)^2} = \lim_{h \to 0} \frac{2x + h}{x^2(x + h)^2} = \frac{2x}{x^2} = \frac{2}{x^3}
\]