Derivative of a Function \( f(x) \) at a Number \( a \)

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \text{ or } f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},
\]
provided that the limit exists. (Note that the former formula is used more often than the latter.)

Example. Using the definition of the derivative at a number, find the slope of tangent line at \( x = -2 \) to the curve \( y = -6x^2 \). What is the equation of the tangent line at this point?

Solution. By definition of the derivative, the slope of the tangent line at \( x = -2 \) is

\[
f'(-2) = \lim_{h \to 0} \frac{f(-2 + h) - f(-2)}{h}
\]

\[
= \lim_{h \to 0} \frac{-6(-2 + h)^2 - (-6(-2)^2)}{h}
\]

\[
= \lim_{h \to 0} \frac{-24 + 24h - 6h^2 + 24}{h}
\]

\[
= \lim_{h \to 0} \frac{h(24 - 6h)}{h} = \lim_{h \to 0} (24 - 6h) = 24
\]

When \( x = -2 \), \( y(-2) = -6(-2)^2 = -24 \), that is, the point of tangency is \((-2, -24)\).

Using the point-slope formula,

\[
y + 24 = 24(x + 2)
\]

\[
y = 24x + 24.
\]
\[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}, \text{ for any } x \text{ for which this limit exists.} \]

Example. Using the definition of the derivative, given the function

\[ f(x) = 3 - \frac{1}{x^2}, \text{ find } f'(x). \]

Solution. By the definition of the derivative,

\[
\begin{align*}
  f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
  &= \lim_{h \to 0} \frac{\left[ 3 - \frac{1}{(x + h)^2} \right] - \left[ 3 - \frac{1}{x^2} \right]}{h} \\
  &= \lim_{h \to 0} \frac{-x^2 + (x + h)^2}{h} \\
  &= \lim_{h \to 0} \frac{x^2 (x + h)^2}{h} \\
  &= \lim_{h \to 0} \frac{h(2x + h)}{hx^2(x + h)^2} \\
  &= \lim_{h \to 0} \frac{2x + h}{x^2(x + h)^2} = \frac{2x}{x^2} = \frac{2}{x^3}
\end{align*}
\]